Minimal Taylor algebras

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Abstract

Much of the research on the Constraint Satisfaction Problem (CSP for short) has been driven by the Dichotomy Conjecture of Feder and Vardi that claims that every CSP of a certain kind is either solvable in polynomial time or is NP-complete. The algebraic approach to the CSP developed to tackle the Dichotomy Conjecture has recently led to its resolution independently by Bulatov and Zhuk (both in FOCS 2017). The algebras key to these results are known as Taylor algebras. It has been known that a CSP corresponding to an algebra that is not Taylor is NP-complete. The proofs of Bulatov and Zhuk confirm that if a CSP corresponds to a Taylor algebra, then it is tractable. The two proofs exploit different features of Taylor algebras, but there seem to be certain structural properties that present themselves in both approaches.

Developing a structural theory of Taylor algebras appears to be unattainable using current tools. In fact, none of the proofs work with a general Taylor algebra; in both cases the authors proceed to reducts of Taylor algebras by dropping some of the operations. The reducts differ from one proof to another, which makes it very hard to compare the algorithms.

We propose and investigate a new class of algebras, minimal Taylor algebras. This class is rich: indeed, every minimal Taylor algebra gives rise to a tractable CSP according to the Dichotomy Theorem, and every algebra that gives rise to a tractable CSP can be reduced to a minimal Taylor algebra by removing some of its operations. In particular establishing tractability for minimal Taylor algebras would provide an alternative proof for the Dichotomy Conjecture.

The existing dichotomy proofs can be directly compared in the class of minimal Taylor algebras and the algebras exhibit very strong properties, making the structural theory much more plausible than in the general case. The long term goal (and one of the motivations) for this research is to apply the structural theory of minimal Taylor algebras to simplify and unify the two proofs of the Dichotomy Conjecture, as well as to provide a powerful machinery for studying other CSP-related complexity problems.

Keywords constraint satisfaction problem, algebraic approach, absorption, binary absorption, graph of algebra

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1 Introduction

The Constaint Satisfaction Problem (CSP) has attracted much attention from researchers in various disciplines. One direction of the CSP research has been greatly motivated by the so-called Dichotomy Conjecture of Feder and Vardi [29, 30]. It was discovered by Jeavons et al. [22, 35, 36] that the CSP and especially the Dichotomy Conjecture has deep connections to Universal Algebra. This line of research become known as the algebraic approach to the CSP. The algebraic approach has been explored further [2, 8, 10, 16, 19] and turned out to be very useful in the study of the complexity of the CSP. Although originally aimed at the standard decision CSPs, variants of the approach proved to be an efficient tool in other types of constraint problems including Quantified CSP [13, 26, 44], the Counting CSP [17, 21], some optimization problems, e.g. the Valued CSP [38] and robust approxiability [7], related promise problems such as "approximate coloring" and the Promise CSP [15, 25], and many others.

The algebraic approach has deep impact on the study of the CSP and universal algebra; it led to major advances in the classification of the complexity (and other properties) of many constraint problems. On the algebraic side, the connection to CSP revitalized the research of finite universal algebras and shifted its focus to problems related to structural properties of algebras that can be exploited algorithmically. The shift resulted in settling some old open questions e.g. [1] and produced many new open problems which seem to be important and natural e.g. [3].

A number of natural classes of algebras directly correspond to CSPs with structural properties that can be used by algorithms. These include e.g. the "few subpowers" algebras [12], or algebras of "bounded width" [6, 14, 19]. Investigation of one such class, the class of Taylor algebras, culminated in two independent proofs, by Bulatov [20] and Zhuk [42], of the Dichotomy Conjecture.

Taylor algebras are those that possess a so-called Taylor term – a term operation satisfying a certain set of equations. They give rise, once the Dichotomy Conjecture is confirmed, to the CSPs solvable in polynomial time. Properties of these algebras played a crucial role in establishing the Dichotomy Conjecture, the two existing proofs of the conjecture [20, 42] discover and exploit different aspects of the structure of Taylor algebras. Both proofs are (understandably) very focused on the task at hand; they solve CSP instances, and do not develop any wide and coherent theory of Taylor algebras. In particular, the structural properties they use are quite different, and the way the arguments go makes them difficult to compare or to find common features. For instance, both proofs start with an arbitrary Taylor algebra, but immediately proceed to work with a certain derivative algebra, a reduct of the original algebra, which is also a Taylor algebra. Since the reducts are defined differently, their structural properties are often incomparable.

There is however a subclass of Taylor algebras, on which the approaches [20] and [42] follow more or less the same lines. This is the class of *minimal Taylor algebras*, that is, algebras such that any their proper reduct fails to have a Taylor term (and so it gives rise to an NP-hard CSP). On the CSP side, minimal Taylor algebras determine the richest, and yet tractable, CSP templates. It can be shown that every Taylor algebra has a minimal Taylor reduct, which implies that establishing tractability for minimal Taylor algebras would imply the Dichotomy Conjecture. On the other hand, minimal Taylor algebras have a number of beneficial properties. For instance, there are only finitely many of them (up to term equivalence) over a set of fixed size. The most important such property for this paper is that the algorithms from [20, 42] cannot change a minimal Taylor algebra, as such algebras have no proper Taylor reducts, and so one can track and compare their work on CSPs over minimal Taylor algebras.

The class of minimal Taylor algebras was first defined by Brady in the context of his investigation of *minimal algebras of bounded width* [14]. In this paper we initiate a systematic study of this class. Our ultimate goal is to understand its structure and to find a unified and hopefully simplified proof of the Dichotomy Conjecture. Apart from clear benefits to Universal Algebra, such a study will shed light on the structure of not only minimal Taylor algebras but general Taylor algebras as well and this can be used in the algebraic approach to CSP which goes well beyond the scope of the Dichotomy Conjecture. Finally, the initial results presented in this paper indicate that minimal Taylor algebras are unusually well behaved and are much more accessible than the class of all Taylor algebras.

To date three quite different approaches to the structure of Taylor algebras exist in connection with the CSP. The main technical tool in the proof by Bulatov [20] are graphs of relational structures (or the corresponding algebras of polymorphisms) whose edges may have one of the three types. Various structural features of such graphs, such as connectivity and connected components formed by edges of certain types, play an important role in this approach. Zhuk in [42] uses special types of absorbing subuniverses and so-called centers of algebras. A defining feature of his proof is a four-way classification of algebras used to split the proof into cases. Finally there is a more general notion of absorption and absorbing universes [2, 8] that led to a number of strong results in the CSP research.

The main contributions of this paper is a unification of the three approaches in the case of minimal Taylor algebras. Firstly, in Section 3, we introduce and prove several basic properties of minimal Taylor algebras. Next, in Section 4, 5, and 6 we give shorter and unified proofs of the basics of Bulatov's and Zhuk's approaches. The results on the connectivity of graphs of algebras, unified operations defining the type of edges used in [20], as well as the 4-way classification from [42] now follow (for minimal Taylor algebras) from a single result, Theorem 4.2, which is of independent interest. We then show, e.g. Theorem 7.6 or Theorem 7.19, how the structural features defined in all the three approaches relate in the case of minimal Taylor algebras. We conclude the paper by proving, in Section 8 strong structural properties, of minimal Taylor algebras omitting edges of certain types. In particular, we prove that such classes usually coincide with classes of algebras studied before and/or of independent interest.

2 Preliminaries

2.1 Algebras

Algebras, i.e. structures with purely functional signature, will be denoted by boldface capital letters (e.g., $\mathbf{A}$) and their universes typically by the same letter in the plain font (e.g., A). The basic general algebraic concepts, such as subuniverses, subalgebras, products, and quotients modulo congruences are used in the standard way (see, e.g. [11]). We use $B \leq \mathbf{A}$ to mean that B is a subuniverse of $\mathbf{A}$. By a *subpower* or a *compatible relation* we mean a subuniverse (or a subalgebra) of a finite power. The set of all compatible relations is denoted $\mathrm{Inv}(\mathbf{A})$. The subuniverse (or the subalgebra) of $\mathbf{A}$ generated by a set $X \subseteq A$ is denoted $\mathrm{Sg}_{\mathbf{A}}(X)$ or $\mathrm{Sg}_{\mathbf{A}}(x_1, \ldots, x_n)$ when $X = \{x_1, \ldots, x_n\}$.

All theorems in this paper concern algebras that are finite and *idempotent*, that is, $f(x, x, \ldots, x) = x$ for every operation f in the algebra and every element x of the universe; the reason being that finite idempotent algebras are exactly those of interest for the CSP over finite templates. **We do not explicitly mention this assumption in the statements of theorems or definitions.**

A *(function) clone* is a set of operations $\mathscr{C}$ on a set A which contains all the projections proj_i^n (the n-ary projection to the i-th coordinate) and is closed under composition, i.e., $f(g_1, \ldots, g_n) \in \mathscr{C}$ whenever $f \in \mathscr{C}$ is n-ary and $g_1, \ldots, g_n \in \mathscr{C}$ are all m-ary, where $f(g_1, \ldots, g_n)$ denotes the operation defined by $(x_1, \ldots, x_m) \mapsto f(g_1(x_1, \ldots, x_m), \ldots, g_n(x_1, \ldots, x_m))$. An arity-preserving mapping ζ from $\mathscr{C}$ to $\mathscr{D}$ is a *clone homomorphism* if the projections are mapped to projections (onto the same coordinate) and ζ preserves composition, i.e., $\zeta(f(g_1, \ldots, g_n)) = \zeta(f)(\zeta(g_1), \ldots, \zeta(g_n))$. By $\mathrm{Clo}(\mathbf{A})$ ($\mathrm{Clo}_n(\mathbf{A})$, respectively), we denote the clone of all term operations (all n-ary term operations, respectively) of $\mathbf{A}$. An algebra $\mathbf{B}$ is a *reduct* of $\mathbf{A}$ if they have the same universe $A = B$ and $\mathrm{Clo}(\mathbf{B}) \subseteq \mathrm{Clo}(\mathbf{A})$. Algebras $\mathbf{A}$ and $\mathbf{B}$ are *term-equivalent* if each of them is a reduct of the other.

2.2 Relations

A relation on A is a subset of A^n, but we often work with more general "multisorted" relations $R \subseteq A_1 \times A_2 \times \cdots \times A_n$. We call such an R *proper* if $R \neq A_1 \times \cdots \times A_n$ and *nontrivial* if it is nonempty and proper. Tuples are written in boldface and components of $\mathbf{x} \in A_1 \times \cdots \times A_n$ are denoted $x_1, x_2, \ldots$. Both $\mathbf{x} \in R$ and $R(\mathbf{x})$ are used to denote the fact that $\mathbf{x}$ is in R. The projection of R onto the coordinates $i_1, \ldots, i_k$ is denoted $\mathrm{proj}_{i_1,\ldots,i_k}(R)$. The relation R is *subdirect*, denoted $R \subseteq_{sd} A_1 \times \cdots \times A_n$, if $\mathrm{proj}_i(R) = A_i$ for each i. We call R *redundant*, if there exist coordinates $i \neq j$ such that $\mathrm{proj}_{ij}(R)$ is a graph of bijection from A_i to A_j; otherwise R is *irredundant*.

We say that a set of relations $\mathcal{R}$ *pp-defines* S if S can be defined from $\mathcal{R}$ by a primitive positive formula with parameters, that is, using the existential quantifier, relations from $\mathcal{R}$, the equality relation, **and the singleton unary relations**. Recall that the set of subpowers of an algebra is closed under pp-definitions.

For binary relations we write $-R$ instead of R^{-1} and $R + S$ for the relational composition of R and S, that is $R + S = \{(a, c) : (\exists b)\, R(a, b) \wedge R(b, c)\}$. For a unary relation B we write $B + S$ to denote the set $\{c : (\exists b)\, B(b) \wedge S(b, c)\}$ and if B is a singleton we write $b + S$ instead of $\{b\} + S$. Also, we set $R - S = R + (-S) = R \circ S^{-1}$. A relation $R \subseteq A \times B$ is *linked* if $(R - R) + (R - R) + \cdots + (R - R)$ is equal to A^2. In other words, R is connected when viewed as a bipartite graph between A and B (with possible isolated vertices). The *left center* of $R \subseteq A \times B$ is the set $\{a \in A : a + R = B\}$. If R has a nonempty left center, it is called *left central*. Right center and right central relations are defined analogically. A relation is central if it is left central and right central. Note that $R+S$, $-R$, and the left (right) center of R are pp-definable from $\{R, S\}$.

2.3 Taylor algebras and abelian algebras

Now we recall a central concept of the algebraic theory of CSPs (and Universal Algebra), Taylor algebra. Instead of defining this notion using Taylor terms, we give another standard, semantic definition.

Definition 2.1. *An (idempotent, finite) algebra* $\mathbf{A}$ *is a* Taylor algebra *if no quotient of a subalgebra of a power of* $\mathbf{A}$ *is a two-element algebra whose every operation is a projection.*

This concept provides the borderline between the NP–complete and tractable CSPs: if the (idempotent) algebra of polymoprhisms of a core CSP template is Taylor, then the CSP is tractable (this is the difficult part in the dichotomy results [20, 42], and otherwise it is NP-complete [22].

There are many equivalent characterizations of Taylor algebras, for instance the powers can be dropped from the definition [23]. In this paper, we will often take advantage of the characterization by means of *cyclic operations* [5].

Theorem 2.2. *An algebra* $\mathbf{A}$ *is Taylor if and only if, for each prime* $p > |A|$, $\mathbf{A}$ *has a term operation* t *of arity* p *which is* cyclic, *that is, for any* $\mathbf{x} \in A^n$,

$$t(x_1, x_2, \ldots, x_p) = t(x_2, \ldots, x_p, x_1).$$

We will be working with cyclic operations often, and often use the following easy properties. If t, s are cyclic operations of arities p and q then *star composition* of t and s defined as

$$t\big(s(x_1, x_{p+1}, \ldots, x_{qp-p+1}), \ldots, s(x_p, x_{2p}, \ldots, x_{qp})\big)$$

is a cyclic operation of arity pq. Moreover if f is any operation of arity p then the *cyclic composition* of t and f

$$t\big(f(x_1, \ldots, x_p), f(x_2, \ldots, x_p, x_1), \ldots, f(x_p, x_1, \ldots, x_{p-1})\big)$$

is a cyclic operation of arity p.

Several further types of operations are significant for this paper:

- *Semilattice operation* is a binary operation $\vee$ which is commutative, idempotent, and associative.
- *Majority operation* is a ternary operation m satisfying $m(x, x, y) = m(x, y, x) = m(y, x, x) = x$ (for any x, y in the universe).
- *Mal'cev operation* is a ternary operation p satisfying $p(y, x, x) = p(x, x, y) = y$.

Any algebra with a semilattice, or majority, or Mal'cev operation is Taylor. The following algebras are particularly important for our purposes:

- *Two-element semillatice*: a two-element set together with one of the two semilattice operations, e.g., $(\{0, 1\}; \vee)$ where $\vee$ is the maximum operation,
- *Two-element majority algebra*: a two element set together with the unique majority operation, e.g., $(\{0, 1\}; \mathrm{maj})$. We also use maj_p, for odd p, to denote the p-ary majority operation on $\{0, 1\}$, that is, $\mathrm{maj}(\mathbf{a}) = 1$ iff the majority of a_i's is 1.
- *Affine Mal'cev algebra*: a set together with the Mal'cev operation $x-y+z$, where $+$ and $-$ is computed with respect to a fixed abelian group structure on the universe, e.g., $(\{0, 1, \ldots, p-1\}; x - y + z \pmod p)$.

The last example falls into a larger class of algebras, which is also significant in the algebraic theory of CSPs, so called *abelian algebras*.

Definition 2.3. *An algebra* $\mathbf{A}$ *is* abelian *if the diagonal* $\Delta_A = \{(a, a) : a \in A\}$ *is a block of a congruence of* $\mathbf{A}^2$.

As an example, for an affine Mal'cev algebra, a (unique) congruence satisfying the definition is the congruence α defined by

$$((x_1, x_2), (y_1, y_2)) \in \alpha \text{ iff } x_1 - x_2 = y_1 - y_2.$$

Note that an abelian algebra does not need to be Taylor, e.g., an algebra with no operations is such, unless it is one-element.

2.4 Semilattice, majority, and abelian edges

Now we introduce the central concepts used in Bulatov's approach to the CSP.

Definition 2.4. *Let* $\mathbf{A}$ *be an algebra. A pair* $(a, b) \in A^2$ *is a* weak edge *if there exists a proper congruence* θ *on* $\mathrm{Sg}_{\mathbf{A}}(a, b)$ *(a* witness *for the edge) such that one of the following happens:*

- *(*weak semilattice edge*) There is a term operation* $f \in \mathrm{Clo}_2(\mathbf{A})$ *acting as a semilattice operation on* $\{a/\theta, b/\theta\}$ *with top element* b/θ*.*
- *(*weak majority edge*) There is a term operation* $m \in \mathrm{Clo}_3(\mathbf{A})$ *acting as a majority on* $\{a/\theta, b/\theta\}$*.*
- *(*weak abelian edge*) The algebra* $\mathrm{Sg}_{\mathbf{A}}(a, b)/\theta$ *is abelian.*

An weak edge (a, b) *is called* an edge *if for some maximal congruence* θ *witnessing the weak edge and every* $a', b' \in A$ *such that* $(a, a'), (b, b') \in \theta$*, we have* $\mathrm{Sg}_{\mathbf{A}}(a', b') = \mathrm{Sg}_{\mathbf{A}}(a, b)$*.*

A witnessing congruence θ for a weak edge (a, b) necessarily separates a and b, i.e., $(a, b) \notin \theta$, since each congruence block of an idempotent algebra is a subuniverse. Also observe that any maximal congruence of $\mathrm{Sg}_{\mathbf{A}}(a, b)$, which contains a witnessing congruence for the edge (a, b), witnesses the weak edge as well.

Note that if (a, b) is a weak edge (or an edge) of majority or abelian type, then so is (b, a). Therefore the direction of an edge matters only for the semilattice edges.

2.5 Absorbing sets and centers

Finally, we introduce absorbing subuniverses and centers, central concepts in, e.g., [42].

Definition 2.5. *Let* $\mathbf{A}$ *be an algebra and* $B \subseteq A$*. We call* B *an* n*-absorbing set of* $\mathbf{A}$ *if there is a term operation* $t \in \mathrm{Clo}_n(\mathbf{A})$ *such that* $t(\mathbf{a}) \in B$ *whenever* $\mathbf{a} \in A^n$ *and* $|\{i : a_i \in B\}| \geq n - 1$*.*

If, additionally, B *is a subuniverse of* $\mathbf{A}$*, we write* $B \trianglelefteq_n \mathbf{A}$*, or* $B \trianglelefteq \mathbf{A}$ *when the arity is not important.*

We remark that of particular interest for us are subuniverses absorbing with a binary or a ternary term operation.

Definition 2.6. *A subset* $C \subseteq A$ *is a* center *of* $\mathbf{A}$ *if there exists an algebra* $\mathbf{B}$ *(of the same signature) with no nontrivial 2-absorbing subuniverse and* $R \leq_{sd} \mathbf{A} \times \mathbf{B}$ *such that* C *is the left center of* R*. The relation* R *is called a* witnessing *relation. If* $\mathbf{B}$ *can be chosen Taylor, we call* C *a* Taylor center *of* $\mathbf{A}$.

Note that center is necessarily a subuniverse.

3 Minimal Taylor algebras

In some contexts, such as in the CSP, forgetting some term operations (i.e., taking reducts) can only make the problem at hand harder. If, moreover, the algebras we are interested in are Taylor (like in the CSP), it is natural to concentrate on the "hardest" algebras, the minimal Taylor ones. This approach was suggested by Brady in his work on minimal bounded width algebras [14].

Definition 3.1. *An algebra* $\mathbf{A}$ *is called a* minimal Taylor algebra *if it is Taylor but no proper reduct of* $\mathbf{A}$ *is.*

Examples of minimal Taylor algebras include two-element semilattices, two-element majority algebras, and affine Mal'cev algebras. This follows from the description of their term operations: the term operations of the two-element semilattice $(\{0, 1\}; \vee)$ are exactly the operations of the form $x_{i_1} \vee x_{i_2} \vee \dots \vee x_{i_k}$; the term operations of the two-element majority algebra $(\{0, 1\}; \mathrm{maj})$ are exactly the idempotent, monotone (i.e., compatible with the inequality relation $\leq$), and self-dual (i.e., compatible with the disequality relation $\neq$) operations; the term operations of an affine Mal'cev algebra over an abelian group $\mathbf{G}$ are exactly the operations of the form $a_1x_1 + a_2x_2 + \dots + a_nx_n$, where a_i are integers that sum up to one. (Each of the mentioned facts is either simple or follows from [40].)

A more complicated example is the three-element "rock-paper-scissors" algebra $(\{\text{paper}, \text{rock}, \text{scissors}\}; \text{winner}(x, y))$. To see that this algebra is minimal Taylor observe that any term operation behaves on any two element set like a semilattice operation. Hence, the original operation can be obtained by identifying variables in any term operation having at least two non-dummy variables.

It is not immediate from the definitions that each Taylor algebra has a minimal Taylor reduct. Nevertheless, this fact easily follows from the characterization of Taylor algebras by means of cyclic operations. This and all the other results in this subsection were already essentially proved in [14] in the context of minimal bounded width algebras.

Proposition 3.2. *Every Taylor algebra has a minimal Taylor reduct.*

Proof. By Theorem 2.2, every Taylor algebra $\mathbf{A}$ has a cyclic term operation of a prime arity p greater than $|A|$. Consider a minimal clone among all clones generated by a cyclic term operation $t \in \mathrm{Clo}(\mathbf{A})$ of arity p (we have finitely many of them). Since any Taylor reduct of $(A; t)$ has t as a term operation, $(A; t)$ is a minimal Taylor reduct of $\mathbf{A}$. □

Another simple, but important consequence of cyclic operations is the following proposition.

Proposition 3.3. *Let* $\mathbf{A}$ *be a minimal Taylor algebra and* $B \subseteq A$ *be closed under an operation* $f \in \mathrm{Clo}(\mathbf{A})$ *such that* B *together with the restriction of* f *to* B *forms a Taylor algebra. Then* B *is a subuniverse of* $\mathbf{A}$.

Proof. Choose a prime number $p > |A|$ and a cyclic operation $t \in \mathrm{Clo}(\mathbf{A})$ of arity p guaranteed by Theorem 2.2. Since B together with the restriction of f to B forms a Taylor algebra, applying Theorem 2.2 again, there exists a term in the operation symbol f defining an operation $s \in \mathrm{Clo}(\mathbf{A})$ of arity p preserving B which is cyclic on B. Then the cyclic composition h of t and s is a term operation of $\mathbf{A}$ which is cyclic and, moreover, preserves B as t is idempotent and s is cyclic on B. Since $\mathbf{A}$ is minimal Taylor, $\mathrm{Clo}(\mathbf{A}) = \mathrm{Clo}(A; h)$ and B is thus a subuniverse of $\mathbf{A}$. □

The same idea proves the following fact whose proof is in Appendix A.

Proposition 3.4. *Any subalgebra, finite power, or quotient of a minimal Taylor algebra is a minimal Taylor algebra.*

We finish with a consequence of star composition and Theorem 2.2.

Proposition 3.5. *Any term operation of a minimal Taylor algebra* $\mathbf{A}$ *can be obtained by identifying and permuting coordinates (and adding dummy coordinates) of a cyclic term operation of* $\mathbf{A}$.

Proof. Since $\mathbf{A}$ is minimal Taylor, $\mathrm{Clo}(\mathbf{A}) = \mathrm{Clo}(A; t)$ for any cyclic operation $t \in \mathrm{Clo}(\mathbf{A})$ (which exists by Theorem 2.2). The claim now follows by noting that the star composition of cyclic operations is a cyclic operation and that, since t is idempotent, any term operation defined by a term in the symbol t can be defined by star composing t multiple times and then permuting and identifying coordinates. □

4 Edges

In this section we present simple proofs of a core fact for Bulatov's proof of the CSP dichotomy conjecture, the connectivity theorem (Theorem 1 in [18]). The theorem states that for any algebra $\mathbf{A}$, the digraph with vertex set A whose arcs are the edges is connected. The original proof uses advanced universal algebraic machinery, while here we prove it using a number of auxiliary statements with direct and self-contained proofs. We start with a simple lemma.

Lemma 4.1. *Let* $\mathbf{A}$ *be an algebra and assume that there exists a ternary* $R \leq \mathbf{A}^3$ *such that all the binary projections of* R *are equal to* A^2 *and a tuple in* R *is determined by values on any two coordinates. Then* $\mathbf{A}$ *is abelian.*

Proof. We fix $c \in A$ and define the binary relation ρ on A^2 by

$$\rho((x_1, x_2), (y_1, y_2)) = \exists u, v, u', v' : \\ R(u, v, x_1) \wedge R(u, c, x_2) \wedge R(u', v, y_1) \wedge R(u', c, y_2).$$

Since a tuple of R is determined by any two coordinates, $x_1 = x_2$ implies $v = c$, and this implies $y_1 = y_2$, and vise versa. Since any binary projection of R is full, for any (x_1, x_2) we can choose u and then v such that $(u, v, x_1), (u, c, x_2) \in R$. Putting $u' = u$, $y_1 = x_1$, and $y_2 = x_2$, we obtain that ρ is a reflexive relation on A^2. Then the congruence on $\mathbf{A}^2$ generated from ρ has a diagonal block $\Delta_A = \{(a, a) : a \in A\}$, which means that $\mathbf{A}$ is abelian. □

The next step is to show that if an algebra has proper, subdirect subpowers, then one can find subpowers of a very particular shape. A fully self-contained proof of the following theorem can be found in Appendix B.

Theorem 4.2. *Let* $R \subseteq_{sd} A^n$ *be an irredundant proper relation. Then either*

- R *pp-defines* $R' \subseteq_{sd} A^2$ *which is irredundant and proper, or*
- *there exist ternary relations* $R_1, \ldots, R_n \subseteq_{sd} A^3$*:*
  - *binary projections of* R_i *are equal to* A^2*,*
  - *a tuple in* R_i *is determined by values on any two coordinates,*

  and the set $\{R_1, \ldots, R_m\}$ *is inter-pp-definable with* R.

In special cases, e.g. when $\mathbf{A}$ is simple, the binary relations that appear in the first case of previous theorem can be made central. An easy proof of the following proposition can be found in Appendix C.

Proposition 4.3. *Let* $R \subseteq_{sd} A^2$ *be linked and proper. Then* R *pp-defines a subdirect proper central relation on* A *which is symmetric or transitive.*

We proceed to prove the connectivity theorem. The proof uses the following, somewhat unnatural, concept.

Definition 4.4. *Let* $\mathbf{A}$ *be an algebra. By the* connected-by-subuniverses *equivalence, denoted* $\mu_{\mathbf{A}}$*, we mean the smallest equivalence containing all the pairs* (a, b) *such that* $\mathrm{Sg}_{\mathbf{A}}(a, b) \neq A$.

We remark that the equivalence $\mu_{\mathbf{A}}$ is not, in general, a congruence of $\mathbf{A}$.

The following theorem is crucial for the connectivity property. It will be also used in Section 6 to provide an easy proof of Theorem 6.1. The proof is direct, if a bit technical, and can be found in Appendix D.

Theorem 4.5. *Let* $\mathbf{A}$ *be a simple algebra with* $|A| \geq 3$.

1. *the algebra* $\mathbf{A}$ *is abelian, or*
2. *there are no subdirect proper irredundant subpowers of* $\mathbf{A}$ *and there exists a term operation* $t \in \mathrm{Clo}_3(\mathbf{A})$ *such that for any* $(a, b) \notin \mu_{\mathbf{A}}$, $t(a, a, b) = t(a, b, a) = t(b, a, a) = a$*, or*
3. *there exists a proper linked subdirect subuniverse of* $\mathbf{A}^2$ *and there exists* $\mu_{\mathbf{A}}$*-class* A' *such that for every* $a \in A'$, $b \notin A'$ *there is a term operation* $t \in \mathrm{Clo}_2(\mathbf{A})$ *such that* $(\{a, b\}; t)$ *is isomorphic to* $(\{0, 1\}; \vee)$ *via the isomorphism* $a \mapsto 1$, $b \mapsto 0$.

We finally proceed to prove the connectivity theorems.

Theorem 4.6. *The directed graph formed by the weak edges of any algebra is connected.*

Proof. We proceed by the way of contradiction. Let $\mathbf{A}$ be a minimal, with respect to size, counterexample to the claim. If $\mathbf{A}$ has two elements the result follows from the classification of Boolean clones by Post [40]. Thus $\mathbf{A}$ has more than two elements and the relation $\mu_{\mathbf{A}}$ cannot be full, as otherwise the connectedness of $\mathbf{A}$ follows from the connectedness of proper subuniverses.

Suppose that β is a non-trivial congruence on $\mathbf{A}$. By the minimality of $\mathbf{A}$ the congruence blocks of β as well as $\mathbf{A}/\beta$ have connected directed graphs of weak edges. Let $(a/\beta, b/\beta)$ be a weak edge in this last directed graph, witnessed by a congruence θ on $\mathrm{Sg}_{\mathbf{A}/\beta}(a/\beta, b/\beta)$.

Let θ' be θ, but treated as a congruence on $\mathrm{Sg}_{\mathbf{A}}(a, b)$. Note that θ' separates a and b and further $\mathrm{Sg}_{\mathbf{A}}(a, b)/\theta'$ is isomorphic to $\mathrm{Sg}_{\mathbf{A}/\beta}(a/\beta, b/\beta)/\theta$ and thus (a, b) is an weak edge in $\mathbf{A}$ of the same type as $(a/\beta, b/\beta)$ in $\mathbf{A}/\beta$. This implies that $\mathbf{A}$ is connected by weak edges and is a contradiction in case when $\mathbf{A}$ has a non-trivial congruence.

Thus $\mathbf{A}$ is simple, has more than two elements, the relation $\mu_{\mathbf{A}}$ is not full, and each equivalence class of $\mu_{\mathbf{A}}$ is connected by weak edges (again from the minimality of $\mathbf{A}$). We apply Theorem 4.5 to conclude that either $\mathbf{A}$ is abelian and every pair is a weak abelian edge, or every pair $(a, a') \notin \mu_{\mathbf{A}}$ is a weak majority edge, or that there exists a $\mu_{\mathbf{A}}$-block A' such that every pair $a \in A', b \notin A'$ forms a weak semilattice edge (b, a) (and in every case the witnessing congruence is the identity congruence). This concludes the proof. □

Theorem 4.7. *The directed graph formed by the edges of any algebra is connected.*

Proof. Let $\mathbf{A}$ be a minimal counterexample to the theorem. By Theorem 4.6, A is connected by weak edges and it suffices to show that each two elements connected by a weak edge are connected by edges. Let (a, b) be a weak edge in $\mathbf{A}$ and let θ be the maximal (see the discussion after the definition of edges) congruence on $\mathrm{Sg}_{\mathbf{A}}(a, b)$ witnessing the edge. The blocks of θ are connected by edges (by the minimality of $\mathbf{A}$).

Choose a', b' such that $(a, a'), (b, b') \in \theta$ and such that $\mathbf{B} = \mathrm{Sg}_{\mathbf{A}}(a', b')$ is minimal. Note that $\mathrm{Sg}_{\mathbf{A}}(a, b)/\theta$ is isomorphic to $\mathbf{B}/\theta_{|\mathbf{B}}$ and thus $\theta_{|\mathbf{B}}$ is a maximal congruence in $\mathbf{B}$, Thus (a', b') is an edge of the same type as the weak edge (a, b). The theorem is proved. □

Notice that the last two theorems do not require that the algebra is Taylor. The main difference for Taylor algebras is that simple abelian algebras are characterized (as essentially modules over finite rings), by a well-known universal algebraic result sometimes called the fundamental theorem on abelian algebras (see [37] for a discussion).

5 Edges in minimal Taylor algebras

The next theorem says that, in minimal Taylor algebras, every "thick" edge, in the terminology of [18, 19], is automatically a subalgebra, a property which is relatively painful to achieve using the original approach.

Theorem 5.1. *Let (a, b) be a weak edge (semilattice, majority, or abelian) of a minimal Taylor algebra $\mathbf{A}$ and θ a witnessing congruence of $\mathbf{E} = \mathrm{Sg}_{\mathbf{A}}(a, b)$.*

(a) *If (a, b) is a weak semilattice edge, then $\mathbf{E}/\theta$ is term equivalent to a two-element semilattce.*
(b) *If (a, b) is a weak majority edge, then $\mathbf{E}/\theta$ is term equivalent to a two-element majority algebra.*
(c) *if (a, b) is a weak abelian edge, then $\mathbf{E}/\theta$ is term equivalent to an affine Mal'cev algebra.*

Proof. In (a) there is a binary term acting like a semilattice term on $\{a/\theta, b/\theta\}$. By Proposition 3.3 together with Proposition 3.4, the set $\{a/\theta, b/\theta\}$ is a subuniverse of $\mathbf{E}/\theta$ and thus equal to it. By the classification of Post [40] and minimality of $\mathbf{E}/\theta$ we conclude (a). The case of (b) is identical, except the term is a ternary majority. In (c), the fundamental theorem on abelian algebras (see [37]) implies that we have the Mal'cev operation $x - y + z$ for some abelian group. □

For edges we can say a bit more. If (a, b) is an edge witnessed by θ a congruence on $\mathrm{Sg}_{\mathbf{A}}(a, b)$ then $\theta = \mu_{\mathbf{E}}$ and $\mathbf{E}/\theta$ is simple. In particular, for abelian edges, $\mathbf{E}/\theta$ is an affine algebra over a p-element abelian group for some prime p. Moreover, such an $\mathbf{E}$ has a unique maximal congruence as shown in the next proposition. This implies that, by the minimality of $\mathbf{A}$, the type of an edge is unique and so is the direction of a semilattice edge and the prime associated with the abelian group.

Proposition 5.2. *Let (a, b) be an edge in a minimal Taylor algebra. Then $\mathrm{Sg}_{\mathbf{A}}(a, b)$ has a unique maximal congruence. In particular, edges have unique types.*

Proof. Let $\mathbf{E} = \mathrm{Sg}_{\mathbf{A}}(a, b)$ and let θ be the congruence on $\mathbf{E}$ witnessing the edge. If (a, b) is a semilattice or a majority edge, Theorem 5.1 implies that $E = a/\theta \cup b/\theta$. By the definition of minimality, there are no proper subalgebras intersecting both a/θ and b/θ and thus every non-full congruence is below θ.

Assume now that (a, b) is an abelian edge. Suppose, for a contradiction, that α is a congruence incomparable with θ. Define $\mathbf{F} = \{(a'/\theta, a''/\theta) : \alpha(a', a'')\}$ and note that $F \leq_{sd} (\mathbf{E}/\theta)^2$ and that it is linked (since θ is maximal). Since $\mathbf{F}/\theta$ has a Mal'cev term operation, we immediately conclude that $F = (E/\theta)^2$, in particular, $(a/\theta, b/\theta) \in F$. This, however, implies that an α-class intersects both a/θ and b/θ and we arrive at the same contradiction as in the case of a semilattice or majority edge.

From the definition of a minimal Taylor algebra, it follows immediately that the same quotient cannot witness two types of an edge (or two directions for a semilattice edge). □

The structure of semilattice edges is especially simple, as follows from the next lemma.

Lemma 5.3. *Let (a, b) be a weak semilattice edge in a minimal Taylor algebra $\mathbf{A}$ and θ be the witnessing congruence* *of* $\mathbf{E} = \mathrm{Sg}_{\mathbf{A}}(a, b)$. *Let* $c \in E$ *be such that* $(b, c) \in \theta$ *and that the subuniverse* $E_c = \mathrm{Sg}_{\mathbf{A}}(a, c)$ *is minimal in* $\{E_c\}_{c \in b/\theta}$. *Then* $\{a, c\}$ *is a subuniverse of* $\mathbf{A}$*, and the subalgebra with universe* $\{a, c\}$ *is term equivalent to the two-element semilattice with top element* c.

Proof. Fix notation as in the statement of the lemma and additionally let $\mathbf{E}' = \mathrm{Sg}_{\mathbf{A}}(a, c)$, $\theta' = \theta \cap (E')^2$, and $R' = \mathrm{Sg}_{\mathbf{A}^2}((a, c), (c, a))$. Since $\mathbf{E}'/\theta'$ is isomorphic to $\mathbf{E}/\theta$ which is, by Theorem 5.1, term equivalent to the two-element join semilattice with top b/θ we conclude that there exists a pair $(c', c'') \in R' \cap (c/\theta')^2$. Then $(c/\theta') + R'$ contains c'' and a and, by the minimality of E', it is equal to E'.

This in turn implies that for some $c''' \in c/\theta'$ we have $(c''', c) \in R'$ and, since $(a, c) \in R'$, the minimality assumption implies that $\{c\} - R' = E'$. In particular $(c, c) \in R'$, i.e., there is a binary term operation acting on $\{a, c\}$ as a join-semilattice operation with top c. By Proposition 3.3, $\{a, c\}$ is a subuniverse of $\mathbf{A}$, by Proposition 3.4 the subalgebra with this subuniverse is a minimal Taylor algebra, which is clearly term equivalent to a two-element semilattice. □

Corollary 5.4. *Let* (a, b) *be a semilattice edge in a minimal Taylor algebra. Then* $\{a, b\}$ *is a subuniverse of* $\mathbf{A}$*, so* $\mathrm{Sg}_{\mathbf{A}}(a, b) = \{a, b\}$ *and the witnessing congruence is the equality.*

Unfortunately, majority edges do not simplify in a similar way; see Example 5.5. Weaker versions of Lemma 5.3 have been developed by Bulatov (comp. Lemma 12 and Corollary 13 in [18]) to deal with this problem.

Example 5.5. *Let* $A = \{0, 1, 2, 3\}$ *and* α *the equivalence relation on* A *with blocks* $\{0, 2\}$ *and* $\{1, 3\}$. *Define two ternary operations* maj *and* min *on* A *as follows:* maj *is majority and* min *is the third projection on* A/α. *On each of the* α*-blocks* $\{0, 2\}, \{1, 3\}$, *operation* maj *is the first projection, and* min *is the minority operation. Finally, for any* $a, b, c \in A$ *such that* $(b, c) \in \alpha$*, but* $(a, b) \notin \alpha$ *we set* $\mathrm{maj}(a, b, c) = \mathrm{maj}(b, a, c) = c$, $\mathrm{maj}(b, c, a) = a+1 \pmod 4$, $\mathrm{min}(a, b, c) = \mathrm{min}(b, a, c) = c+2 \pmod 4$, $\mathrm{min}(b, c, a) = c + 3 \pmod 4$. *Let* $\mathbf{A} = (A, \mathrm{maj}, \mathrm{min})$. *As is easily seen,* $\mathbf{A}$ *is Taylor, and any pair* (a, b), $a \in \{0, 1\}$, $b \in \{1, 3\}$, *is a weak majority edge, as witnessed by the congruence* α. *It can be verified by straightforward computation (use Universal Algebra Calculator [31]) that for no such pair there is a term operation of* $\mathbf{A}$ *that is majority on* $\{a, b\}$. *Taking any minimal Taylor reduct* $\mathbf{A}'$ *of* $\mathbf{A}$ *we conclude, that in order for the directed edge graph of* $\mathbf{A}'$ *to be connected we need to allow majority edges which are not subuniverses.*

Another important fact for the edge approach is that semilattice, majority, and Mal'cev operations coming from edges can be unified, see Theorem 7 in [18]. Brady observed (personal communication) that this theorem can be proved by a simple argument using cyclic operations. A direct proof of the following fact can be found in Appendix E.

Corollary 5.6. *Every minimal Taylor algebra* $\mathbf{A}$ *has a ternary term operation* f *such that if* (a, b) *is an edge witnessed by* θ *on* $\mathbf{E} = \mathrm{Sg}_{\mathbf{A}}(a, b)$*, then*

- *if* (a, b) *is a semilattice edge, then* $f(x, y, z) = x \vee y \vee z$ *on* $\{a, b\}$ *(where* b *is the top);*
- *if* (a, b) *is a majority edge, then* f *is the majority operation on* $\mathbf{E}/\theta$ *(which has two elements);*
- *if* (a, b) *is an abelian edge, then* $f(x, y, z) = x - y + z$ *on* $\mathbf{E}/\theta$.

6 The four types

A core fact for Zhuk's proof of the CSP dichotomy conjecture, Theorem 4 in [42], shows that each Taylor algebra has one of the four types of interesting subuniverses or quotients. His original proof uses a complicated result, Rosenberg's classification of maximal clones [41]. Here we show that the four-types classification is a simple consequence of results we have already established, in particular, Theorem 4.2.

Recall that an algebra $\mathbf{A}$ is *polynomially complete* if every operation on A can be obtained by substituting elements of A to some coordinates of a term operation of $\mathbf{A}$; equivalently, $\mathbf{A}$ has no proper reflexive (that is, containing all the tuples $(a, a, \ldots, a)$) irredundant subpowers.

Theorem 6.1. *Let* $\mathbf{A}$ *be an algebra, then*

(a) $\mathbf{A}$ *has a nontrivial 2-absorbing subuniverse, or*
(b) $\mathbf{A}$ *has a nontrivial center (which is a Taylor center in the case that* $\mathbf{A}$ *is a Taylor algebra), or*
(c) $\mathbf{A}/\alpha$ *is abelian for some proper congruence* α *of* $\mathbf{A}$*, or*
(d) $\mathbf{A}/\alpha$ *is polynomially complete for some proper congruence* α *of* $\mathbf{A}$.

Proof. Let α be a maximal congruence on $\mathbf{A}$ and consider subpowers of the simple algebra $\mathbf{B} = \mathbf{A}/\alpha$. If every subdirect, irredundant subpower of $\mathbf{B}$ is full, then $\mathbf{B}$ is polynomially complete, i.e., case (d) holds.

Otherwise we apply Theorem 4.2 to an irredundant, proper $R \leq_{sd} \mathbf{B}^n$. If R does not pp-define a subdirect, proper subuniverse of B^2, then it defines at least one ternary relation satisfying the assumptions of Lemma 4.1. Therefore $\mathbf{B}$ is abelian and we are in case (c).

We are left with the case that there exists an irredundant, proper $S \leq_{sd} \mathbf{B}^2$. Since $\mathbf{B}$ is simple, the relation S needs to be linked (see the argument in the proof of Theorem 4.5) and, by Proposition 4.3, S pp-defines a proper and central $T \leq_{sd} \mathbf{B}^2$. Now $T' = \{(a, a') \in A^2 : (a/\alpha, a'/\alpha) \in T\}$ is a proper central subdirect subuniverse of $\mathbf{A}^2$. Either $\mathbf{A}$ has a 2-absorbing subuniverse, i.e., case (a) holds, or T witnesses that $\mathbf{A}$ has a nontrivial center, i.e., we are in case (b). □

Note that the proof gives additional properties in some of the cases: in case (b), the center can be witnessed by a subuniverse of $\mathbf{A}^2$ and, in case (d), the quotient algebra can be required to have no proper subdirect irrendundant subpowers.

Just like the "connectivity" Theorem 4.7, the last theorem as well does not require the algebra to be Taylor. If it is, then centers have additional pleasant properties (see Proposition 7.18). If it is not, we can still say something: either $\mathbf{A}$ has a nontrivial *cube term blocker* (a subuniverse satisfying item (d) in Theorem 7.6) or $\mathbf{A}$ has a nontrivial absorbing subuniverse, see the proof of Lemma 7.15.

Examples of minimal Taylor algebras, for which one of the cases takes place and no other, are (a) a two-element semilattice, (b) a two-element majority algebra, (c) an affine Mal'cev algebra, and (d) the three element rock-paper-scissors algebra discussed in Section 3.

7 Absorption in minimal Taylor algebras

In this section we study properties of absorbing sets and their interaction with other concepts, such as centers or edges, in minimal Taylor algebras. We start with a surprising fact, whose proof is in Appendix F, that absorbing subsets are necessarily subuniverses.

Theorem 7.1. *Let $\mathbf{A}$ be a minimal Taylor algebra and B an absorbing set of $\mathbf{A}$. Then B is a subuniverse of $\mathbf{A}$.*

We devote separate subsections to 2-absorbing sets and n-absorbing sets for $n \geq 3$. In both of them, the following new concept, which extends the standard definition of a clone homomorphism, will be useful.

Definition 7.2. *Let $\mathscr{C}, \mathscr{D}$ be two function clones. We call a relation $\zeta \subseteq \mathscr{C} \times \mathscr{D}$ a* clone relation *if*

- *ζ preserves arities, i.e., if $(f, f') \in \zeta$, then the arities of f and f' are equal,*
- *for every $i, n, i \leq n$ we have $(\mathrm{proj}_i^n, \mathrm{proj}_i^n) \in \zeta$ (where the first proj_i^n is in $\mathscr{C}$ and the second in $\mathscr{D}$), and*
- *if $(f, f') \in \zeta$ of arity n and $(f_1, f_1'), \ldots, (f_n, f_n') \in \zeta$ all of arity k, then $(f(f_1, \ldots, f_n), f'(f_1', \ldots, f_n')) \in \zeta$.*

Note that any arity-preserving relation $\zeta \subseteq \mathscr{C} \times \mathscr{D}$, typically $\zeta = \{(f, f')\}$, generates a clone relation by closing ζ under the projection pairs $(\mathrm{proj}_i^n, \mathrm{proj}_i^n)$ and composition. Let C and D denote the domains of $\mathscr{C}$ and $\mathscr{D}$, respectively. Observe that if $\zeta = \{(f, f'), \mathscr{C}$ is generated by f, and $(D; f')$ is in the variety generated by $(C; f)$, then this clone relation is in fact a clone homomorphism. This typically, happens when $\mathscr{C}$ is the clone of a minimal Taylor algebra and f is its cyclic term operation.

One auxiliary notion will be used to deal with clone relations. The *characteristic function of a subset $B \subseteq A$*, denoted χ_B^A or just χ_B if A is clear from the context, is the function $A \to \{0, 1\}$ such that $\chi_B^A(b) = 1$ iff $b \in B$. We extent χ_B^A component-wise to tuples, so we also have $\chi_B^A : A^n \to \{0, 1\}^n$ for each n.

7.1 Binary absorption

Binary absorbing sets (i.e., 2-absorbing sets) have especially strong properties in minimal Taylor algebras, similarly to semilattice edges. We give several characterizations of these sets in Theorem 7.6 and Propositions 7.8 and 7.7. Before doing so, we prove two implications in Theorem 7.6 separately. The following concept is required: a subset B of $\mathbf{A}$ is asm-*closed* if there is no edge (b, a) such that $b \in B$ and $a \in A \setminus B$.

Lemma 7.3. *Every 2-absorbing set in a minimal Taylor algebra is* asm-*closed.*

Proof. For a contradiction, assume that $B \trianglelefteq_2 \mathbf{A}$ and that (b, a) is a minimal edge such that $b \in B$ and $a \in A \setminus B$. Let θ be the maximal congruence of $\mathbf{C} = \mathrm{Sg}_{\mathbf{A}}(b, a)$ witnessing the (b, a) edge. By the minimality property of an edge, we have $a/\theta \cap B = \emptyset$ (as otherwise B intersects a/θ and b/θ). Then $D = \{c/\theta : c/\theta \cap B \neq \emptyset\}$ is a binary absorbing subuniverse of $\mathbf{C}/\theta$ that does not contain a/θ. This cannot happen in either of the three cases (semilattice, majority, abelian) by the description of term operations given in Section 3. □

We call a coordinate $i \in \{1, \ldots, n\}$ of an n-ary operation f *essential* if $f(\mathbf{a}) \neq f(\mathbf{b})$ for some tuples $\mathbf{a}, \mathbf{b}$ that differ only at the coordinate i.

Lemma 7.4. *Let $\mathbf{A}$ be a minimal Taylor algebra and $B \trianglelefteq_2 \mathbf{A}$. Then for every $f \in \mathrm{Clo}_n(\mathbf{A})$ and every essential coordinate i of f, we have $f(\mathbf{a}) \in B$ whenever $\mathbf{a} \in A^n$ is such that $a_i \in B$.*

As will be seen from the proof, the requirement that i is an essential coordinate can be weakened to requiring only that f is defined from a cyclic operation by a term where the coordinate appears.

Proof. Let g be a binary operation witnessing $B \trianglelefteq_2 \mathbf{A}$ and let $t \in \mathrm{Clo}(\mathbf{A})$ be a cyclic operation of arity, say p. Define $h(x_1, \ldots, x_p)$ as

$$g(\cdots g(g(x_1, x_2), x_3), \ldots x_p)$$

and note that $h(a_1, \ldots, a_i) \in B$ whenever at least one of the a_i's is in B. The same property has the cycle composition, call it s, of t and h. Clearly, s generates the whole clone of $\mathbf{A}$ (in particular, f) and if a variable appearing in a term build from s (in particular, a variable corresponding to an essential coordinate of f) is evaluated to B, then the whole term is. □

The last lemma has an interesting consequence in terms of clone relations.

Corollary 7.5. *Let $\mathbf{A}$ be a minimal Taylor algebra and $t \in \mathrm{Clo}(\mathbf{A})$ be a cyclic operation of arity p. Then the clone relation ζ between $\mathrm{Clo}(\mathbf{A})$ and $\mathrm{Clo}(\{0, 1\}; \vee)$ generated by $(t, \bigvee_{i=1}^p x_i)$ satisfies*

$$\chi_B(g(\mathbf{a})) \geq g'(\chi_B(\mathbf{a}))$$

for any $B \trianglelefteq_2 \mathbf{A}$, any n-ary, $(g, g') \in \zeta$, and any $\mathbf{a} \in A^n$.

Note that the clone relation is a clone homomorphism whenever $\mathbf{A}$ has a semilattice edge. Also note that the inequality is only interesting if the right hand side is 1 and then it tells us that $g(\mathbf{a})$ is in B.

Proof. The inequality is satisfied for the generating pair by Lemma 7.4 and, trivially, for the projection pairs. It remains to check that the inequality is stable under composition. Let $(f, f') \in \zeta$ be n-ary and , $(f_1, f_1'), \dots, (f_n, f_n') \in \zeta$ be k-ary such that the inequality holds for all of them. Pick an arbitrary $\mathbf{a} \in A^k$ and denote $g = f(f_1, \dots, f_n)$, $g' = f'(f_1', \dots, f_n')$. For any $i \in \{1, \dots, n\}$, we have $f_i'(\chi_B(\mathbf{a})) \leq \chi_B(f_i(\mathbf{a}))$, therefore $g'(\chi_B(\mathbf{a})) = f'(f_1'(\chi_B(\mathbf{a})), \dots, f_n'(\chi_B(\mathbf{a})))$ is less than or equal to $f'(\chi_B(f_1(\mathbf{a})), \dots, \chi_B(f_n(\mathbf{a})))$ by the monotonicity of f'. The last expression is less than or equal to $\chi_B(f(f_1(\mathbf{a}), \dots, f_n(\mathbf{a}))) = \chi_B(g(\mathbf{a}))$ by the inequality for (f, f') applied to the tuple $(f_1(\mathbf{a}), \dots, f_n(\mathbf{a}))$, and we are done. □

We are ready to prove the aforementioned characterization of binary absorbing subuniverses.

Theorem 7.6. *The following are equivalent for any minimal Taylor algebra* $\mathbf{A}$ *and a set* $B \subseteq A$.

(a) $B \trianglelefteq_2 \mathbf{A}$.
(b) $R(x, y, z) = B(x) \vee B(y) \vee B(z)$ *is a subuniverse of* $\mathbf{A}^3$.
(c) *For every* $f \in \mathrm{Clo}_n(\mathbf{A})$ *and every essential coordinate* i *of* f, *we have* $f(\mathbf{a}) \in B$ *whenever* $\mathbf{a} \in A^n$ *is such that* $a_i \in B$.
(d) *For every* $f \in \mathrm{Clo}_n(\mathbf{A})$ *there exists a coordinate* i *of* f *such that* $f(\mathbf{a}) \in B$ *whenever* $\mathbf{a} \in A^n$ *is such that* $a_i \in B$.

Moreover, these properties imply that

(e) B *is* asm*-closed.*

Proof. First (a) implies (c) by Lemma 7.4; (c) trivially implies (d) and for (d) to imply (a) it suffices to take any cyclic operation t and the absorption is witnessed by $t(x, y, \dots, y)$.

The implication from (d) to (b) is clear. For (b) implies (a) we first consider a p-ary cyclic term operation t with $p > 2$ and a tuple $\mathbf{a}$ satisfying $a_i \in B$ for all $i < p/3 + 1$. Take cyclic shifts $\mathbf{b}$ and $\mathbf{c}$ of $\mathbf{a}$ so that at least one of a_i, b_i, and c_i is in B for each $i \leq p$. Then $t(\mathbf{a}) = t(\mathbf{b}) = t(\mathbf{c})$ by cyclicity of t and $t(\mathbf{a}) \in B$ by compatibility with R. It follows that $t(x, \dots, x, y, \dots, y)$, with one more x than y, witnesses $B \trianglelefteq_2 \mathbf{A}$.

Finally, Lemma 7.3 states exactly that (a) implies (e). □

Items (a), (c) and (d) admit relational descriptions. A subuniverse B with property (d) is known under the name *cube term blocker* and the relational description is simple.

Proposition 7.7 (Lemma 3.2 in [39])**.** *Let* $\mathbf{A}$ *be an algebra. The relations* $A^n \setminus (A \setminus B)^n$ *is compatible with* $\mathbf{A}$ *for every* n *if and only if for every* $f \in \mathrm{Clo}_n(\mathbf{A})$ *there exists a coordinate* i *of* f *such that* $f(\mathbf{a}) \in B$ *whenever* $\mathbf{a} \in A^n$ *is such that* $a_i \in B$.

A relational description of item (c) is also quite straightforward to prove, see Appendix G.

Proposition 7.8. *Let* $\mathbf{A}$ *be an algebra. The relation* $R(x, y, z) = B(x) \vee (y = z)$ *is a subuniverse of* $\mathbf{A}^3$ *if and only if for every* $f \in \mathrm{Clo}_n(\mathbf{A})$ *and every essential coordinate* i *of* f, *we have* $f(\mathbf{a}) \in B$ *whenever* $\mathbf{a} \in A^n$ *is such that* $a_i \in B$.

A somewhat more complicated relational description of item (a), can be deduced from Lemma 7.16.

We complete this subsection by two corollaries that establish strong interactions of 2-absorbing subuniverses with other subuniverses.

Corollary 7.9. *Let* $\mathbf{A}$ *be a minimal Taylor algebra and* $B \trianglelefteq_2 \mathbf{A}$.

1. *If* $C \leq \mathbf{A}$ *then* $B \cup C \leq \mathbf{A}$.
2. *If* $C \trianglelefteq \mathbf{A}$ *with a witnessing operation* f, *then*
   a. $B \cup C \trianglelefteq \mathbf{A}$ *by* f, *and*
   b. $B \cap C \neq \emptyset$ *and* $B \cap C \trianglelefteq \mathbf{A}$ *by* f.

Proof. For (1) consider the result of applying an operation $f \in \mathrm{Clo}(\mathbf{A})$ to a tuple $\mathbf{a}$. If $a_i \in C$ for all the essential coordinates i of f, then the result is in C (as $C \leq \mathbf{A}$), and if $a_i \in B$ for an essential i, then the result is in B by item (c) in Theorem 7.6.

The argument for (2.a) is similar. Indeed, if a tuple $\mathbf{a}$ has all but one entry in $B \cup C$, then $f(\mathbf{a}) \in B$ in case that $a_i \in B$ for an essential i, or $f(\mathbf{a}) \in C$ in the other case (as $C \trianglelefteq \mathbf{A}$ by t). For (2.b) observe that the operation f witnessing a proper absorption $C \trianglelefteq \mathbf{A}$ has at least two essential coordinates (while the case $C = A$ is trivial). Then $B \cap C \trianglelefteq \mathbf{A}$ follows again from item (c) in Theorem 7.6, and $B \cap C$ is nonempty since it contains $f(c, \dots, c, b, c, \dots, c)$ for any $b \in B$ and $c \in C$ (where b is at an essential coordinate). □

Corollary 7.10. *Every minimal Taylor algebra* $\mathbf{A}$ *has a unique minimal 2-absorbing subalgebra* $\mathbf{B}$. *Moreover,* $\mathbf{B}$ *does not have any nontrivial 2-absorbing subuniverse.*

Proof. By Corollary 7.9, the intersection $\mathbf{B}$ of all 2-absorbing subalgebras of $\mathbf{A}$ is 2-absorbing (and, in fact, any binary term operation of $\mathbf{A}$ whose both coordinates are essential can be taken as a witness). For the second part, note that if $\emptyset \neq C \trianglelefteq_2 \mathbf{B}$, then both absorptions $C \trianglelefteq_2 \mathbf{B} \trianglelefteq_2 \mathbf{A}$ can be witnesses by the same operation f and then $f(f(f(x, y), x), f(f(y, x), y))$ witnesses $\mathbf{C} \trianglelefteq_2 \mathbf{A}$, so $C = B$. □

7.2 Ternary (and higher arity) absorption

In this subsection we present several implications between centers, absorbing sets, and subsets closed under abelian, semilattice, or majority edges.

Analogously to asm-closed sets, we call a subset B of $\mathbf{A}$ as*-closed* (m*-closed*, respectively) if there is no semilattice or abelian edge (majority edge, respectively) (b, a) such that $b \in B$ and $a \in A \setminus B$. The first lemma is an analogue of Lemma 7.3.

Lemma 7.11. *Every absorbing set in a minimal Taylor algebra is* as*-closed.*

Proof. The proof is almost the same as for Lemma 7.3, the only difference being that the subuniverse D is absorbing (instead of 2-absorbing) and therefore a majority edge can appear. □

The next theorem, an analogue of Corollary 7.5, may be of independent interest and will be applied to prove that ternary absorbing sets in a minimal Taylor algebra are centers (Lemma 7.14) which are, in case they are m-closed, even binary absorbing (Proposition 7.13).

Theorem 7.12. *Let* $\mathbf{A}$ *be a minimal Taylor algebra and* $p > |A|$ *a prime. Then there exists a* p*-ary cyclic operation* $t \in \mathrm{Clo}(\mathbf{A})$ *such that the clone relation* ζ *between* $\mathrm{Clo}(\mathbf{A})$ *and* $\mathrm{Clo}(\{0,1\};\mathrm{maj})$ *generated by* (t, maj_p) *satisfies*

$$\chi_B(g(\mathbf{a})) \geq g'(\chi_B(\mathbf{a}))$$

for any $B \trianglelefteq_3 \mathbf{A}$*, any* n*-ary* $(g, g') \in \zeta$*, and any* $\mathbf{a} \in A^n$*.*

As before, if $\mathbf{A}$ has majority edge, then ζ is a clone homomorphism.

Proof. The reasoning in Corollary 7.5 shows that it is enough to verify the inequality for a generator since we used only the monotonicity of the operations in the second clone on $\{0,1\}$.

Given a set $\mathcal{B}$ of 3-absorbing subuniverses and a cyclic operation $t \in \mathrm{Clo}_p(\mathbf{A})$, such that (t, maj_p) satisfies the inequality for every $B \in \mathcal{B}$, we will find another cyclic term operation s which will still work for any $B \in \mathcal{B}$ but also for a new 3-absorbing subuniverse $C \trianglelefteq_3 \mathbf{A}$. The claim will then follow by induction since we can start with the empty $\mathcal{B}$ and any p-ary cyclic term operation of $\mathbf{A}$.

Let ζ be the clone relation generated by (t, maj_p), let f be a witness for $C \trianglelefteq_3 \mathbf{A}$, and let f' be a ternary operation in $\mathrm{Clo}(\{0,1\};\mathrm{maj})$ such that $(f, f') \in \zeta$ (such an f' exists since the cyclic operation t generates the whole clone of $\mathbf{A}$). Finally, let h be the p-ary term operation of $\mathbf{A}$ defined from f by the same term as a term defining maj_p from maj (the latter term exists by the structure of the majority clone described in Section 3) and let s be the cyclic composition of t and h. Our aim is to verify the inequality for (s, maj_p) and any $B \in \mathcal{B} \cup \{C\}$.

Observe that, by definition of h, the pair (h, maj_p) satisfies the inequality for $B = C$ and then so does (s, maj_p) by the definition of s and $B \leq \mathbf{A}$. We thus further concentrate on the case $B \in \mathcal{B}$. If $f' = \mathrm{maj}$, then from $(f, f') \in \zeta$ we get $(h, \mathrm{maj}_p) \in \zeta$ (using the term defining h from f). In this case, (h, maj_p) and then (s, maj_p) satisfies the inequality also for $B \in \mathcal{B}$ and we are done.

Otherwise f' is a projection (again by the structure of the majority clone), so $(f, f') \in \zeta$ gives $(h, \mathrm{proj}_i^p) \in \zeta$ for some i. Consequently, $\chi_B(a_i) = \mathrm{proj}_i^p(\chi_B(\mathbf{a})) \leq \chi_B(h(\mathbf{a}))$. Therefore, if $\mathrm{maj}_p(\chi_B(\mathbf{a})) = 1$, i.e., a majority of entries of $\mathbf{a}$ is in B, then so is a majority of elements $h(\mathbf{a}^j)$, where $\mathbf{a}^j$ denotes the j-th cyclic shift of $\mathbf{a}$. Since $(t, \mathrm{maj}_p) \in \zeta$ we get $\chi_B(t(h(\mathbf{a}^0), h(\mathbf{a}^1), \ldots)) = \chi_B(s(\mathbf{a})) = 1$ and the inequality is verified. □

Proposition 7.13. *Let* $\mathbf{A}$ *be a minimal Taylor algebra. If* $B \trianglelefteq_3 \mathbf{A}$ *and* B *is* m*-closed, then* $B \trianglelefteq_2 \mathbf{A}$*.*

Proof. Fix B and $\mathbf{A}$ as in the statement and take a cyclic operation t and clone relation ζ provided by the previous theorem.

First note that for any $a \in A$, $b \in B$, and $f \in \mathrm{Clo}_2(\mathbf{A})$ either $f(a,b) \in B$ or $f(b,a) \in B$. Indeed, assuming $f(a,b) \notin B$ and taking f' so that $(f, f') \in \zeta$, we get $f'(0,1) = 0$ (by the inequality in the theorem); this in turn implies that $f'(1,0) = 1$ (as the only binary operations in the majority clone are the projections) and, using the inequality again, $f(b,a) \in B$.

In order to establish the proposition it now suffices, to find, for any $a \in A, b \in B$, a term operation f_{ab} such that both $f_{ab}(a,b)$ and $f_{ab}(b,a)$ are in B. Indeed, we can then gradually build a term operation working for all pairs (a,b), $(c,d), \ldots$ (with $b, d, \dots \in B$) following the pattern

$$f_{f_{ab}(c,d) f_{ab}(d,c)}(f_{ab}(x,y), f_{ab}(y,x))$$

to eventually obtain a witness for 2-absorption.

Take $a \in A \setminus B$ and $b \in B$, let $\mathbf{C} = \mathrm{Sg}_{\mathbf{A}}(a,b)$, and consider the partition $\{C \cap B, C \setminus B\}$. If t acts like majority of arity p on this partition, then the partition defines a congruence modulo which $\mathbf{C}$ is a two element majority algebra – a weak edge. Taking $a' \in C \setminus B$ and $b' \in C \cap B$ with minimal $\mathrm{Sg}_{\mathbf{A}}(a', b')$ yields a majority edge leaving B, a contradiction.

Therefore there is a tuple $\mathbf{a}$ with elements of B in minority such that $t(\mathbf{a}) \in B$. Take $f_i \in \mathrm{Clo}_2(\mathbf{A})$ such that $f_i(a,b) = a_i$. Now $g(x,y) = t(f_1(x,y), \ldots, t_p(x,y))$ is a binary term operation such that $g(a,b) \in B$ (by construction) and $g(b,a) \in B$ by the inequality for $(t, \mathrm{maj}_p) \in \zeta$ (since a majority of arguments of t is in B by the second paragraph), completing the proof. □

The next lemma shows that 3-absorbing sets are necessarily centers.

Lemma 7.14. *Let* $\mathbf{A}$ *be a minimal Taylor algebra and* $C \trianglelefteq_3 \mathbf{A}$*. Then* C *is a Taylor center of* $\mathbf{A}$*.*

Proof. Take a cyclic operation t provided by Theorem 7.12. Clearly $A \times \{0\} \cup B \times \{1\}$ is a subuniverse of $(A; t) \times (\{0,1\}; \mathrm{maj}_p)$. As $\mathrm{Clo}(A; t) = \mathrm{Clo}(\mathbf{A})$ and $\mathrm{Clo}(\{0,1\}; \mathrm{maj}_p)$ is the clone of the majority algebra on $\{0,1\}$, which does not have any proper 2-absorbing subuniverse, we are done. □

Our next goal is to prove the converse – that a center (or a singleton absorbing subuniverse) in a minimal Taylor algebra is necessarily 3-absorbing. Two known facts are useful in the proof. The first one, implicit in [5], is that a center is necessarily absorbing.

Lemma 7.15. *Let C be a Taylor center of an algebra $\mathbf{A}$ (not necessarily Taylor). Then $C \trianglelefteq \mathbf{A}$.*

Comments on the proof. Let $R \leq_{sd} \mathbf{A} \times \mathbf{B}$ witness that C is a center. Since $\mathbf{B}$ is Taylor and has no non-trivial 2-absorbing subuniverse, $\mathbf{B}$ has no cube term blocker (Lemma 3.4 in [9]). An algebra with no cube term blocker has a transitive term operation (Lemma 2.7 in [9]). The corresponding operation on $\mathbf{A}$ then witnesses $C \trianglelefteq \mathbf{A}$ (see the final part of the proof of Theorem 2.11 in [5]). □

The second fact is a characterization of absorption by means of so called B-essential relations (see Proposition 2.14 in [4]).

Lemma 7.16. *Let $\mathbf{A}$ be an algebra and $B \leq \mathbf{A}$. Then $B \trianglelefteq_n \mathbf{A}$ if and only if for every $\mathbf{a}^1, \ldots, \mathbf{a}^n \in A^n$ such that $\mathbf{a}^i_j \in B$ for $i \neq j$ we have $\mathrm{Sg}_{\mathbf{A}^n}(\mathbf{a}^1, \ldots, \mathbf{a}^n) \cap B^n \neq \emptyset$.*

The last bit is the following observation which, in fact, isolates a significant property of centers exploited in [42]. Its proof is in Appendix H.

Lemma 7.17. *Let C be a center in $\mathbf{A}$ and let $b \in A \setminus C$. Then (b, b) is* not *in the subuniverse of $\mathbf{A}^2$ generated by $(\{b\} \times C) \cup (C \times C) \cup (C \times \{b\})$.*

The goal ("center ⇒ 3-absorbing") can now be proved using the proof-idea of Lemma 7.10 in [43], details are in Appendix H.

Proposition 7.18. *Let $\mathbf{A}$ be an algebra, $B \trianglelefteq \mathbf{A}$, and*

(a) *B be a center of $\mathbf{A}$, or*
(b) *$|B| = 1$ and $\mathbf{A}$ be minimal Taylor.*

Then $B \trianglelefteq_3 \mathbf{A}$.

Theorem 7.19. *Let $\mathbf{A}$ be a minimal Taylor algebra and $B \subseteq A$. Then, in the following list, (a) ⇔ (b) ⇔ (c) ⇒ (d) ⇒ (e). Moreover, if $B = \{b\}$, then (d) ⇒ (c).*

(a) *B is a Taylor center of $\mathbf{A}$.*
(b) *The relation $R(x, y) = B(x) \vee B(y)$ is a subuniverse of $\mathbf{A}^2$.*
(c) *$B \trianglelefteq_3 \mathbf{A}$.*
(d) *$B \trianglelefteq \mathbf{A}$.*
(e) *B is* as*-closed.*

Proof. A combination of Lemma 7.15 and Proposition 7.18(a) gives that that (a) implies (c), (c) implies (a) by Lemma 7.14, (c) trivially implies (d), and (d) implies (e) by Lemma 7.11.

Now (c) implies (b) by Theorem 7.12 as the relation R is obviously compatible with the operation t provided by the theorem. On the other hand (b) implies (c) by a reasoning similar to that in Theorem 7.6: we first consider a p-ary cyclic term operation t with $p > 2$ and a tuple $\mathbf{a}$ satisfying $a_i \in B$ for all $i < p/2 + 1$. Take a cyclic shift $\mathbf{b}$ such that at least one of a_i, b_i is in B for each $i \leq p$. Then $t(\mathbf{a}) = t(\mathbf{b})$ by cyclicity of t and $t(\mathbf{a}) \in B$ by compatibility with R. It follows that $t(x, \ldots, x, y, \ldots, y, z, \ldots, z)$, with the number of x's, y's and z's different by at most 1 witnesses $B \trianglelefteq_3 \mathbf{A}$.

For $B = \{b\}$, (d) implies (c) by Proposition 7.18. □

The following two examples show that (d) does not imply (c) and (e) does not imply (d) even for $B = \{b\}$.

Example 7.20. *Consider the algebra $\mathbf{A} = (\{0, 1, 2\}, m)$ where m is the majority operation such that $m(a, b, c) = a$ whenever $|\{a, b, c\}| = 3$. This algebra is minimal Taylor because m generates a minimal clone (see [27]). The set $C = \{0, 1\}$ is an absorbing subuniverse of $\mathbf{A}$ as witnessed by the 4-ary operation $m(m(m(x_1, x_2, x_3), x_2, x_4), x_3, x_4)$. However, C is not a center of $\mathbf{A}$ since for any potentially witnessing relation $R \leq_{sd} \mathbf{A} \times \mathbf{B}$ the subuniverse $D = 2 + R \leq \mathbf{B}$ satisfies $m(D, C, C) \subseteq D$ (as $m(2, 1, 0) = 2$) and $m(C, D, D) \subseteq D$ (as $m(0, 2, 2) = 2$), so $m(x, y, y)$ witnesses that D is a 2-absorbing subuniverse of $\mathbf{B}$.*

Example 7.21. *Consider the algebra $\mathbf{A} = (\{0, 0', 1\}, m)$ where m is the majority operation on $\{0, 0'\}$, $\sigma = \{0, 0'\}^2 \cup \{1\}^2$ is a congruence such that $\mathbf{A}/\sigma \cong (\{0, 1\}, x + y + z)$, and $m(a, 1, 1) = m(1, a, 1) = m(1, 1, a) = 0'$ for every $a \in \{0, 0'\}$. Since $\mathbf{A}/\sigma \cong (\{0, 1\}, x + y + z)$, the subuniverse $\{0\}$ is not absorbing. Note that $R = \{(0, 0), (0, 0'), (0', 0'), (0, 1), (0', 1)\} \in$* Inv($\mathbf{A}$)*, then every operation from* Clo($\mathbf{A}$) *preserving $\{0, 1\}$ also preserves $R' = R \cap (A \times \{0, 1\}) = \{(0, 0), (0, 1), (0', 1)\}$ and $S = \{(0, 0, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0)\}$. Then the formula*

$$\exists x', y', z' : S(x', y', z') \wedge R'(x, x') \wedge R'(y, y') \wedge R'(z, z')$$

defines $\{0, 0'\}^3 \setminus \{0'\}^3$, which cannot be preserved by a majority on $\{0, 0'\}$. This proves that $\{0, 1\}$ cannot be a subuniverse in any Taylor reduct of $\mathbf{A}$, therefore, $\{0\}$ is as*-closed.*

If we require $\{b\}$ to be closed under weak abelian edges then the implication (e)⇒(d) holds (see Appendix H for the proof).

Proposition 7.22. *Let $\mathbf{A}$ be a minimal Taylor algebra and $b \in B$. If there are no semilattice or weak abelian edges coming from b then $\{b\} \trianglelefteq \mathbf{A}$.*

As it is shown in the next example, for bigger domains even this is not true.

Example 7.23. *Consider the algebra $\mathbf{A} = (\{0, 1, 2\}, m)$ where m is the majority operation such that $m(a, b, c) = 2$ whenever $|\{a, b, c\}| = 3$. This algebra is minimal Taylor because m generates a minimal clone (see [27]). Every pair of distinct elements forms a subuniverse and it is a majority edge. So there are no semilattice or (weak) abelian edges. However, the (*as*-closed) subuniverse $\{0, 1\}$ is not absorbing because of the compatible relation $R = \{0, 1\}^2 \setminus \{(0, 0), (1, 1)\}$. Indeed, if f is a witnessing operation, then the following is an R-walk from 0 to 1 within {0,1} of even length: $f(0, \ldots, 0)$, $f(1, \ldots, 1, 2)$, $f(0, \ldots, 0, 1)$, $\ldots$, $f(0, \ldots, 0, 1, 1,)$, $\ldots$, $f(1, 1, \ldots, 1)$.*

We also remark that as-closed set is not necessarily a subuniverse, as witnessed by the 4-element majority algebra $(\{0, 1, 2, 3\}, m)$ where $m(a, b, c) = 0$ whenever $|\{a, b, c\}| = 3$.

Corollary 7.24. *Let* $\mathbf{A}$ *be a minimal Taylor algebra and* $B, C \trianglelefteq_3 \mathbf{A}$.

1. $B \cup C \leq \mathbf{A}$
2. *If* $B \cap C \neq \emptyset$ *then* $B \cap C \trianglelefteq_3 \mathbf{A}$.
3. *If* $B \cap C = \emptyset$ *then* $B^2 \cup C^2$ *is a congruence on the algebra with universe* $B \cup C$ *and the quotient is term-equivalent to a two-element majority algebra.*

Proof. All three items follow directly from Theorem 7.12. The term t provided by the theorem applied to arguments from $B \cup C$ returns an element of the the set that is represented more often, which proves (1) and (3). Any same term $t(x, \dots, x, y, \dots, y, z, \dots, z)$ with the number of x's, y's and z's different by at most one witnesses $B, C \trianglelefteq_3 \mathbf{A}$ and thus $B \cap C \trianglelefteq_3 \mathbf{A}$ in case (2). □

8 Omitting types

In this section we consider classes of algebras whose graph only contains edges of certain types. We say that an algebra is a-*free* if it has no abelian edges. More generally, an algebra is x-*free* or is xy-*free*, where x, y ∈ {(a)ffine, (m)ajority, (s)emilattice} if it has no edges of type x (of types x, y). It turns out that within minimal Taylor algebras these "omitting types" conditions are often equivalent to important properties of algebras. Recall that the properties of "having bounded width" and "having few subpowers" characterize the applicability of the two basic algorithmic ideas in the CSP – local propagation algorithms [6, 24] and finding a generating set of all solutions [12, 34]. These can be captured by the existence of term operations satisfying certain equations, see [10]. Proofs of the following six theorems can be found in Appendix J.

Theorem 8.1. *The following are equivalent for any minimal Taylor algebra* $\mathbf{A}$.

(a) $\mathbf{A}$ *has bounded width.*
(b) $\mathbf{A}$ *is* a-*free.*

Theorem 8.2. *The following are equivalent for any minimal Taylor algebra* $\mathbf{A}$.

(a) $\mathbf{A}$ *has few subpowers.*
(b) $\mathbf{A}$ *is* s-*free.*
(c) *No subalgebra of* $\mathbf{A}$ *has a nontrivial 2-absorbing subuniverse.*

For the remaining omitting-single-type condition, m-freeness, we do not provide a natural condition in terms of equations. Nevertheless, it can be characterized by means of absorption.

Theorem 8.3. *The following are equivalent for any minimal Taylor algebra* $\mathbf{A}$.

(b) $\mathbf{A}$ *is* m-*free.*
(c) *Every subalgebra of* $\mathbf{A}$ *has a unique 3-minimal absorbing subuniverse.*
(d) *If* $\mathbf{C} \trianglelefteq_3 \mathbf{B} \leq \mathbf{A}$ *then* $\mathbf{C} \trianglelefteq_2 \mathbf{B}$.

We conclude this section with a sequence of theorems characterizing algebras with only one type of edges.

Theorem 8.4. *The following are equivalent for any minimal Taylor algebra* $\mathbf{A}$.

(a) $\mathbf{A}$ *has a Mal'cev term operation.*
(b) $\mathbf{A}$ *is* sm-*free.*
(c) *No subalgebra of* $\mathbf{A}$ *has a nontrivial absorbing subuniverse.*
(d) *No subalgebra of* $\mathbf{A}$ *has a nontrivial center.*

Theorem 8.5. *The following are equivalent for any minimal Taylor algebra* $\mathbf{A}$.

(a) $\mathbf{A}$ *has a* wnu operations *i.e. an operation satisfying* $f(y, x, \dots, x) = f(x, y, x, \dots, x) = \dots = f(x, \dots, x, y)$ *of every arity greater than or equal to 2.*
(b) $\mathbf{A}$ *is* am-*free.*

Theorem 8.6. *The following are equivalent for any minimal Taylor algebra* $\mathbf{A}$.

(a) $\mathbf{A}$ *has a majority term operation.*
(a') $\mathbf{A}$ *has a near unanimity term operation.*
(b) $\mathbf{A}$ *is* as-*free.*

9 Conclusion

We have introduced the concept of minimal Taylor algebras and used it to significantly unify, simplify, and extend the two main algebraic approaches to the CSP – via colored edges, and via absorption and centers. We believe that the theory started in this paper will help in attacking further open problems in computational complexity of CSP-related problems and Universal Algebra. There are, however, many directions which call for further exploration.

First, several technical questions naturally arise from the presented results: Is there an analogue to Example 5.5 for abelian edges? Are all the items in Theorem 7.6 equivalent? The presented proof of Proposition 7.22 is not simple or self-contained; is there a simpler, self-contained proof? Are the equivalent characterizations in Theorem 8.3 equivalent to "every subalgebra has a unique minimal absorbing (rather than 3-absorbing) subuniverse"? Can the condition (a) in Theorem 8.5 be improved to a obtain a 2-semilattice operation?

Second, Z. Brady in [14] delivered an impressive collection of result about a smaller class of algebras, the minimal bounded width ones, "almost" providing a complete classification. Can such a detailed analysis be made also for minimal Taylor algebras?

Third, both CSP dichotomy proofs [20, 43] require and develop more advanced Commutator Theory [32, 37] concepts and results, while in this paper we have merely used some fundamental facts about the basic concept, the abelian algebra. Is it possible to develop our theory in this direction as well, potentially providing sufficient tools for the dichotomy result?

Finally, there is yet another, older, and highly developed theory of finite algebras, the Tame Congruence Theory started in [33]. What are the connections to the theory initiated in this paper?

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A Proof of Proposition 3.4

Proposition 3.4. *Any subalgebra, finite power, or quotient of a minimal Taylor algebra is a minimal Taylor algebra.*

Proof. For finite powers the claim follows from the definition of a power. Let $\mathbf{A}$ be a minimal Taylor algebra and $\mathbf{B}$ be its subalgebra or quotient. We choose a prime number $p > |A|$ and a p-ary cyclic term operation t of $\mathbf{A}$. Using Theorem 2.2 and Proposition 3.2 we find $s \in \mathrm{Clo}(\mathbf{A})$ such that B together with the corresponding term operation $s^{\mathbf{B}}$ of $\mathbf{B}$ is minimal Taylor. Then the cyclic composition h of t and s is a cyclic operation on $\mathbf{A}$ and the corresponding $h^{\mathbf{B}}$ coincides with $s^{\mathbf{B}}$. Since $\mathbf{A}$ is minimal Taylor, we have $\mathrm{Clo}(\mathbf{A}) = \mathrm{Clo}(A; h)$ and therefore $\mathrm{Clo}(\mathbf{B}) = \mathrm{Clo}(B; h^{\mathbf{B}}) = \mathrm{Clo}(B; s^{\mathbf{B}})$, which completes the proof. □

B Proof of Theorem 4.2

We start with a technical lemma.

Lemma B.1. *Let $R \subseteq_{sd} A^n$ be a relation, and $I \subseteq n$ be a maximal set of coordinates such that $\mathrm{proj}_I(R)$ is the full product $A^{|I|}$. Then every tuple in R is determined by its projection to I, or R pp-defines $R' \subseteq_{sd} A^2$ which is irredundant and proper,*

Proof. Let R be a counterexample minimal with respect to the arity. In particular, no relation pp-definable from R, such as the projection to a subset of variables, pp-defines subdirect, irredundant, and proper binary relation.

The relation R needs to be irredundant, otherwise the projection forgetting one of the redundant coordinates is a counterexample to the lemma of smaller arity. Next, $|I|$ has to be $n-1$, because otherwise the lemma holds for every projection S of R on any $|I|+1$ coordinates containing I, let j be the additional coordinate position. Relation S does not pp-define a subdirect, irredundant and proper binary relation, so by the minimality of I for any $\mathbf{b} \in S$ the value a_j is determined by the remaining coordinates, and that also holds for R. Without loss of generality assume $I = \{1, \ldots, n-1\}$. Consider $\mathrm{proj}_{1,\ldots,n-2,n} R$. By the minimality of R it has to be full. Indeed, otherwise the first $n-2$ coordinates determine the n-th coordinate, contradicting the assumption that R is a minimal counterexample.

Since R is a counterexample there are elements $a \neq a'$ and tuples $(a_1, \ldots, a_{n-1}, a), (a_1, \ldots, a_{n-1}, a') \in R$. Also, as R is not full, $(c_1, \ldots, c_n) \notin R$ for some tuple. Set $T = \{(b_1, \ldots, b_{n-1}) : (b_1, \ldots, b_{n-1}, c_n) \in R\}$; we show that either T or $\mathrm{proj}_{1,\ldots,n-2,n} R$ is a smaller counterexample, thus obtaining a contradiction. Indeed, the relation T is proper, as it does not contain $(c_1, \ldots, c_{n-1})$. Also, $\mathrm{proj}_{1,\ldots,n-2} T$ is the full relation, because otherwise $\mathrm{proj}_{1,\ldots,n-2,n} R$ would be a proper relation, and therefore a smaller counterexample. It remains to show that the values of the first $n-2$ coordinate positions of T do not determine the last one. To this end we consider an auxiliary binary relation S given by $S = \{(a, b) : (a_1, \ldots, a_{n-2}, a, b) \in R\}$. This relation is subdirect, as both $\mathrm{proj}_I R$ and $\mathrm{proj}_{1,\ldots,n-2,n} R$ are full relations. Relation S is also irredundant, because $(a_{n-1}, a), (a_{n-1}, a') \in S$. By the assumptions about R, relation S cannot be proper. Therefore the tuples $(a_1, \ldots, a_{n-2}, a, c_n), (a_1, \ldots, a_{n-2}, a', c_n) \in R$, implying that $(a_1, \ldots, a_{n-2}, a), (a_1, \ldots, a_{n-2}, a') \in T$. □

We are now in a position to prove Theorem 4.2, which reproduce here for reader's convenience.

Theorem 4.2. *Let $R \subseteq_{sd} A^n$ be an irredundant proper relation. Then either*

- *R pp-defines $R' \subseteq_{sd} A^2$ which is irredundant and proper, or*
- *there exist ternary relations $R_1, \ldots, R_m \subseteq_{sd} A^3$ such that:*
  - *binary projections of R_i are equal to A^2,*
  - *a tuple in R_i is determined by values on any two coordinates,*

  and the set $\{R_1, \ldots, R_m\}$ is inter-pp-definable with R.

Proof. First, we argue that R pp-defines some binary or ternary proper and irreducible relations. Let R' be a proper irreducible relation of minimal arity pp-definable by R. If R' is binary or ternary, we are done. Otherwise observe that the projection of R' on any proper set of coordinates is the full relation. Let $(a_1, \ldots, a_k) \notin R'$. Consider the relation S given by

$$S = \{(x_1, \ldots, x_k - 1) : (x_1, \ldots, x_{k-1}, x_{k-1}, a_k) \in R'\}.$$

This relation is proper, as $(a_1, \ldots, a_k) \notin R'$. It is also subdirect, because every binary projection of R' is the full relation. If S is redundant, say, for $i, j \in \{1, \ldots, k-1\}$ it holds that $\mathrm{proj}_{ij} S$ is the graph of bijection, then $\mathrm{proj}_{ijk} R'$ is a proper relation, a contradiction with the choice of R'.

If R pp-defines a binary, proper, and irreducible relation, the first item from the conclusion of the theorem holds. So, suppose that such a binary relation cannot be defined. Let

$R_1, \dots, R_m \subseteq_{sd} A^3$ be all the proper, ternary, irredundant and subdirect relations in the clone. Any binary projection of each R_i is the full relation, and by Lemma B.1, any tuple from R_i is determined by any of its two entries. It remains to prove that the set $\{R_1, \dots, R_m\}$ pp-defines R.

We show, by induction on the arity, that any at least ternary irredundant subdirect relation pp-definable from $R_1, \dots, R_m, R$ is also pp-definable from $R_1, \dots, R_m$. We proceed by contradiction, let $S \subseteq_{sd} A^l$ be a counterexample of minimal arity, that is, $R_1, \dots, R_m, R$ pp-define S, but $R_1, \dots, R_m$ do not. By Lemma B.1 there is $I \subseteq \{1, \dots, l\}$, such that any tuple $(a_1, \dots, a_l) \in S$ is determined by its projection on I and $\mathrm{proj}_I S$ is the full relation. Assume $I = \{1, \dots, k\}$. If $k+1 \neq l$, then

$$S(x_1, \dots, x_k) = \bigwedge_{j=k+1}^{l} \mathrm{proj}_{I \cup \{j\}} S(x_1, \dots, x_k, x_j).$$

By the induction hypothesis every $\mathrm{proj}_{I \cup \{j\}} S$ is pp-definable from $R_1, \dots, R_m$, hence, so is S, a contradiction with the choice of S. Thus we may assume that $k+1 = l$.

Claim. Each projection $S' = \mathrm{proj}_J S$ on $J \subseteq \{1, \dots, l\}$ with $|J| < l$ is full.

Indeed, it is the case if $J \subseteq I$ by the choice of I. Therefore $l \in J$, and, as $\mathrm{proj}_{J \setminus \{l\}} S'$ is the full relation, by Lemma B.1 every tuple from S' is determined by its projection to $J \setminus \{l\}$. This means that S can obtained from S' by extending the tuples from S' in an arbitrary way, and this a pp-definition of S from S'. By induction hypothesis S' is pp-definable from $R_1, \dots, R_m$, and we obtain a contradiction with the choice of S.

Since S is proper, there is $(c_1, \dots, c_l) \notin S$. Set $T = \{(a, b, c) : (c_1, \dots, c_{l-3}, a, b, c) \in S\}$. This relation is proper, since $(c_{l-2}, c_{l-1}, c_l) \notin T$. By the Claim above $\mathrm{proj}_{1,2}(T) = \mathrm{proj}_{1,3}(T) = \mathrm{proj}_{2,3}(T) = A^2$. Thus, T is one of the R_i's. Consider relations U and U' given by

$$U(x_1, \dots, x_l, y) = S(x_1, \dots, x_l) \wedge T(y, x_{l-1}, x_l).$$

and

$$\begin{aligned} U'(x_1, \dots, x_l, y) \quad = \quad & \mathrm{proj}_{1,\dots,l-2,l+1} U(x_1, \dots, x_{l-2}, y) \\ & \wedge T(y, x_{l-1}, x_l). \end{aligned}$$

We show that that they are identical. This will imply the result, because, as is easily seen, $S = \mathrm{proj}_{1,\dots,l} U$, and U' is pp-definable from $R_1, \dots, R_m$, as $\mathrm{proj}_{1,\dots,l-2,l+1}(U)$ is by the induction hypothesis.

It is not hard to see that $U' \subseteq U$. Next, note that $U'' = \mathrm{proj}_{1,\dots,l-2,l+1} U$ is not full, since $U(c_1, \dots, c_{l-3}, a, b, c, d)$ imply that $d = a$. On the other hand, $\mathrm{proj}_{1,\dots,l-2} U$ is full. Therefore by Lemma B.1 for any $(a_1, \dots, a_l, a) \in U$ the value a is determined by $a_1, \dots, a_{l-2}$.

Take a tuple $(a_1, \dots, a_l, a) \in U'$; since $\mathrm{proj}_{1,\dots,l-1} S$ is the full relation, there is $(a_1, \dots, a_{l-1}, c, d) \in U$ for some $c, d \in A$. Since $(a_1, \dots, a_{l-2}, a) \in \mathrm{proj}_{1,\dots,l-2,l+1} U$, and in this relation the last value is determined by the first $l-2$ ones, we have $d = a$. Again by Lemma B.1 the third coordinate of the relation T is determined by the first two ones. Therefore, as we have $T(d, a_{l-i}, c)$ from the definition of U, $T(a, a_{l-i}, a_l)$ from the definition of U', and $d = a$, we also obtain $c = a_l$. Thus, $(a_1, \dots, a_l, a) \in U$ completing the proof. □

C Proof of Proposition 4.3

In this section we prove Proposition 4.3 that we state here again.

Proposition 4.3. *Let $R \subseteq_{sd} A^2$ be linked and proper. Then R pp-defines a subdirect proper central relation on A which is symmetric or transitive.*

We start with an auxiliary lemma.

Lemma C.1. *Let $R \subseteq_{sd} A^2$ be linked and such that the right center of R is empty. Then R pp-defines a subdirect proper central relation on A which is symmetric.*

Proof. First of all we show that R pp-defines a proper $S \subseteq_{sd} A^2$ such that $S - S = A^2$ and the right center of S is empty. If $R - R = A^2$ we can take R for S. Otherwise, as R is linked, $(R-R) + (R-R) + \dots + (R-R) = A^2$ for a sufficiently long composition. Let $S = (R-R) + (R-R) + \dots + (R-R)$ so that $S \neq A^2$ but $S + S = A^2$. Since $R - R$ is symmetric, so is S, i.e. $S = -S$, and thus $S - S = S + S = A^2$. If S has non-empty right center we have obtained a subdirect proper central relation that is symmetric, as required in the lemma. So suppose the right center of S is empty.

Let k be the maximum number such that such that every k-element subset $C \subseteq A$ has a common neighbor $b \in A$, that is, $C \subseteq b - S$. Since the right center of S is empty, $k < |A|$. Also, $k \geq 2$; indeed, if there is $a \in A$ that has only one neighbour b, then, since $S - S = A^2$, $b - S = A$, a contradiction with the assumption that S has the empty right center. Take a set $D = \{d_1, \dots, d_{k+1}\} \subseteq A$ with no common neighbor and define a relation T by

$$T(x, y) = (\exists z) S(x, z) \wedge S(y, z) \wedge S(d_1, z) \wedge \dots \wedge S(d_{k-1}, z).$$

As is easily seen, each of $d_1, \dots, d_{k-1}$ is in the left center of T. Moreover T is symmetric and therefore it is also subdirect. Finally, as as $(d_k, d_{k+1}) \notin S$, T is proper. Thus T is the required central relation. □

Proof of Proposition 4.3. If left or right center of R is empty we apply Lemma C.1 to R itself, or to $-R$, and the result follows. So, let R be central. We also assume, without loss of generality, that the left center of R contains the maximal number of elements among central, proper and subdirect relations pp-definable from R.

Consider the sequence of subdirect relations $R_0 = R$, $R_{i+1} = R_i + R_i$. As is easily seen by induction, every R_i is subdirect and central. Indeed, $R_0 = R$ satisfies these conditions, suppose so does R_i. Then for any $a \in A$ there is

$b, c \in A$ such that $(a, b), (b, c) \in R_i$, implying $(a, c) \in R_{i+1}$ and $\mathrm{proj}_1 R_{i+1} = A$. The equality $\mathrm{proj}_2 R_{i+1}$ is similar. If a belongs to the left center of R_i, i.e. $(a, b) \in R_i$ for each $b \in A$, then, in particular, $(a, a) \in R_i$. Therefore $(a, b) \in R_{i+1}$ for all $b \in A$, and a belongs to the left center of R_{i+1}. A similar argument shows that the right center of R is a subset of the right center of each R_i. Note that this implies that the left centers of all the proper relations R_i are equal: they all contain the left center of R, and, by the choice of R to have a largest left center, they have to be equal to that of R.

For some N it holds that $R_N = R_{N+}$, choose the smallest number with this property. If $R_N \neq A^2$ then it is the desired proper central and transitive relation on A. If $R_N = A^2$, consider $S = R_{N-1}$. It is a subdirect, proper central relation such that $S + S = A^2$.

Next, let B be the right center of S, we consider two cases: either $B + S = A$ or $B + S \neq A$. (Note that B is the **right** center, which implies $B - S = A$, but not necessarily $B + S = A$. So the latter case is possible.)

Case 1. $B + S = A$.

Consider $S' = (S \cap -S)$. This relation is proper, because S is proper is proper and is symmetric by construction. It is also subdirect, as $S + S = A^2$ implies that for every a there is b such that $S(a, b)$ and $S(b, a)$. Finally, S' is also linked. Indeed, note that, since B is the right center, $B^2 \subseteq S$, and so $B^2 \subseteq S'$. Also, the assumption $B + S = A$ implies that for any $a \in A$ there is $b \in B$ such that $(b, a) \in S$. On the other hand, $(a, b) \in S$, because b belongs to the right center. Therefore $(a, b) \in S'$, implying together with $B^2 \subseteq S'$ that S' is linked.

If S' is central, then we are done. Otherwise, since S' is symmetric, its right center is empty, and we use Lemma C.1 to obtain a symmetric central relation.

Case 2. $B + S \neq A$.

We will derive a contradiction that shows that this case is impossible. Let $A = \{a_1, \dots, a_n\}$, and for $j \geq 0$, let the relation T_j be given by

$$T_j(x, y) = (\exists z) S(x, z) \wedge S(z, y) \wedge \bigwedge_{i=1}^{j} S(a_i, z).$$

Clearly, $T_0 = A^2$ and T_n is not even subdirect. Indeed, in the latter case let $(a, b) \in T$, then the value of z in the pp-definition above belongs to B, the right center. As $B + S \neq A$, there is $c \in A$ such that $(z, c) \notin S$ for any feasible choice of z, witnessing that $c \notin \mathrm{proj}_2 T$. Therefore there is j such that $T_{j-1} = A^2$, and $T_j \neq A^2$. We will show that T_j is central and has strictly larger left center than S, which contradicts the choice of R.

By the definition of T_j we have $(a_j, b) \in T_{j-1}$ if and only if $T_j(a_j, b)$, therefore $\{a_j\} + T_j = A$. This implies that $\mathrm{proj}_2(T_j) = A$ and that a_j is in the left center of T_j. Note that every element in the left center of S is in the left center of T_j. Indeed, if a is in the left center of S, then, by the symmetricity of S, it is also in the right center of S. Therefore for any choice of b in $T(a, b)$, the value of z can be set to a, proving that $(a, b) \in T$. Note also that a_j does not belong to the left center of S, because this would imply that $T_{j-1} = T_j$. Since S has non-empty right center, for any $a \in A$, choose the value of z in the definition of T to be from the right center. Then $(a, z), (a_1, z), \dots, (a_j, z) \in S$, and a value for y can be chosen with $(z, y) \in S$, implying that $\mathrm{proj}_1(T_j) = A$.

Thus T_j is proper, subdirect, central and pp-definable from R. However, its left center is a superset of the left center of R and contains a_j that is not in the left center of R. A contradiction with the choice of R.

The proposition is proved. □

D Proof of Theorem 4.5

Theorem 4.5. *Let $\mathbf{A}$ be a simple algebra with $|A| \geq 3$.*

1. *the algebra $\mathbf{A}$ is abelian, or*
2. *there are no subdirect proper irredundant subpowers of $\mathbf{A}$ and there exists a term operation $t \in \mathrm{Clo}_3(\mathbf{A})$ such that for any $(a, b) \notin \mu_{\mathbf{A}}$, $t(a, a, b) = t(a, b, a) = t(b, a, a) = a$, or*
3. *there exists a proper linked subdirect subuniverse of $\mathbf{A}^2$ and there exists $\mu_{\mathbf{A}}$-class A' such that for every $a \in A', b \notin A'$ there is a term operation $t \in \mathrm{Clo}_2(\mathbf{A})$ such that $(\{a, b\}; t)$ is isomorphic to $(\{0, 1\}; \vee)$ via the isomorphism $a \mapsto 1$, $b \mapsto 0$.*

Let $\mathbf{A}$ be as in the statement. Note that if $\mu_{\mathbf{A}}$ is full, the claim boils down to Theorem 4.2 and Lemma 4.1. Therefore, in the remaining part of the proof we assume that $\mu_{\mathbf{A}}$ is not full.

Let $\mathscr{C}$ be the set of all subdirect relations in $\mathrm{Inv}(\mathbf{A})$. We consider cases depending on the relations in $\mathscr{C}$.

D.1 Every irredundant relation in $\mathscr{C}$ is full

In order to prove (2) we consider $\mathbf{R} = \mathrm{proj}_I(\mathbf{A}^{A^3})$ where $I = \{(a_1, a_2, a_3) : \exists a, b \ \{a_1, a_2, a_3\} = \{a, b\} \wedge \neg\mu_{\mathbf{A}}(a, b)\}$ and let S be the subuniverse of $\mathbf{R}$ generated by $\mathbf{d}^1, \mathbf{d}^2, \mathbf{d}^3$ defined by $d^i_{(a_1, a_2, a_3)} = a_i$. Note, that if S contains a tuple $\mathbf{d}$ such that $d_{(a_1, a_2, a_3)}$ is the element which appears in majority in (a_1, a_2, a_3) we have the operation required in (2). The subpower S can be redundant; suppose, e.g., that $\mathrm{proj}_{(a_1, a_2, a_3), (a'_1, a'_2, a'_3)} S$ is a graph of a bijection. Note that, in this case, the position of a non-repeating element in (a_1, a_2, a_3) and in (a'_1, a'_2, a'_3) must be the same and we can assume, without loss of generality, that $(a_1, a_2, a_3) = (a, b, b)$ and $(a'_1, a'_2, a'_3) = (a', b', b')$. The fact that $\mathrm{proj}_{(a,b,b),(a',b',b')} S$ is a graph of bijection implies that, for every term operation t, if $t(a, b, b) = t(b, a, b) = t(b, b, a) = b$ then $t(a', b', b') = t(b', a', b') = t(b', b', a') = b'$. This means that we can consider $S^1 = \mathrm{proj}_J(S)$ where $J = I \setminus \{(a', b', b'), (b', a', b'), (b', b', a')\}$ instead of S. We continue removing redundant coordinates in this way to arrive at a subpower S^i which is irredundant and therefore, by our assumption, full. In particular, we obtain a term operation acting as the majority on the evaluations corresponding to coordinates remaining in S^i and, by construction, also on all other evaluations required by (2).

D.2 There is an irredundant, proper, binary $R \in \mathscr{C}$

In this case we prove (3), in a number of claims. We start with a basic observation: if $R \leq_{sd} \mathbf{A}^2$ and $(a,b), (a',b)$ for $(a,a') \notin \mu_{\mathbf{A}}$ then b is in the right center of R (since $\mathrm{Sg}_{\mathbf{A}}(a,a') = A$).

Claim 1. *Every irredundant, proper $R \leq_{sd} \mathbf{A}^2$ is central.*

Proof. Since $\mathbf{A}$ is simple, R needs to be linked (otherwise the relation "being linked to" on the left side of R would define a proper congruence of $\mathbf{A}$). Therefore, since $\mu_{\mathbf{A}}$ is not the full relation, we have $(a,b), (a',b) \in R$ for $(a,a') \notin \mu_{\mathbf{A}}$ which implies that b is in the right center of R. The proof for left center is symmetric. □

Claim 2. *Let $R \leq_{sd} \mathbf{A}^2$ be irredundant and proper. If a and a' are in the left (right) center of R then $(a,a') \in \mu_{\mathbf{A}}$.*

Proof. Suppose not. Then for every $b \in A$ we have $(a,b), (a',b) \in R$ which implies that b is the right center. This cannot happen in a proper R. □

Claim 3. *Let $R \leq_{sd} \mathbf{A}^2$ be irredundant and proper. If a is in the left center of R and a' in the right center then $(a,a') \in \mu_{\mathbf{A}}$.*

Proof. Suppose, for a contradiction, that a is in the left center of R, a' in the right center and $(a,a') \notin \mu_{\mathbf{A}}$. Since $(a,a), (a',a') \in R$ then, for every $b \in A$, we have $(b,b) \in R$.

Take a'' such that $(a'',a') \notin \mu_{\mathbf{A}}$ then a'' is in the left center of R, as both (a'',a'') and (a'',a') are in R. This implies, by the previous claim, that $(a'',a) \in \mu_{\mathbf{A}}$.

In particular we conclude that $\mu_{\mathbf{A}}$ has two equivalence blocks, and that every a'' such that $(a,a'') \in \mu_{\mathbf{A}}$ is in the left center. By symmetry we get $R = (a/\mu_{\mathbf{A}} \times A) \cup (A \times a'/\mu_{\mathbf{A}})$, but then $R \cap -R$ is a proper congruence (recall $|A| > 2$) on a simple algebra $\mathbf{A}$, a contradiction. □

Claim 4. *Let $R \leq_{sd} \mathbf{A}^2$ be irredundant and proper. There is an $\mu_{\mathbf{A}}$ block B such that $R \subseteq B \times A \cup A \times B$.*

Proof. Fix a in the left center of R and a' in the right; by the previous claim $(a,a') \in \mu_{\mathbf{A}}$. We will show, that if $(b,c) \in R$ then $(b,a) \in \mu_{\mathbf{A}}$ or $(c,a') \in \mu_{\mathbf{A}}$, which proves the claim with $B = a/\mu_{\mathbf{A}} = a'/\mu_{\mathbf{A}}$. Indeed, if $(b,a) \notin \mu_{\mathbf{A}}$ then, as $(b,c), (a,c) \in R$ and c is in the right center of R and thus $(c,a') \in \mu_{\mathbf{A}}$. □

Claim 5. *Let $R \leq_{sd} \mathbf{A}^2$ be irredundant and proper and $S \leq_{sd} \mathbf{A}^2$ redundant. If B is the block defined by the previous claim for R, then $B + S = B$.*

Proof. Suppose not and let $(a,a') \in S$ with $a \in B$ and $a' \notin B$. Then a is in the left center of $R+S$ while a' in the right center of $R+S$ contradicting Claim 3 (the relation $R+S$ is clearly irredundant and proper). □

Claim 6. *Let $R, S \leq_{sd} \mathbf{A}^2$ be irredundant and proper. If B, C are the blocks defined by the Claim 4 for R, S respectively, then $B = C$.*

Proof. Suppose, for a contradiction, that $B \neq C$. Let b, b' be the elements of the right and left centers of R, respectively, and similarly c, c' for S. Let $T = R \cap S$, and note that $(b,c'), (c,b') \in T$. As $(b,c) \notin \mu_{\mathbf{A}}$ and $(c',b') \notin \mu_{\mathbf{A}}$, the relation T is subdirect in A^2.

Since $B \cap C = \emptyset$ the relation T has no center, and thus by Claim 1 needs to be redundant. But then $(B+T) \cap C \neq \emptyset$ which contradicts the previous claim. □

To finish the proof we take a proper, irredundant binary relation provided by the case we are in and set A' to be the block defined for it by Claim 4.

Take any $a' \in A', b \notin A'$ and let $R = \mathrm{Sg}_{\mathbf{A}^2}((a',b),(b,a'))$. The relation R cannot be redundant as $b \in A' + R$ would contradict Claim 5. Thus, by Claim 1 and Claim 6, there is $a'' \in A'$ in the left center of R. Since $(b,a'), (a'',a') \in R$ we conclude that a' is in the right center of R i.e. $(a',a') \in R$ and this case is done as witnessed by the operation generating (a',a') from the generators (a',b), (b,a').

D.3 The remaining case

By Theorem 4.2 in the remaining case there is at least one ternary relation, with all binary projections full, and such that a projection to each two coordinates determines the tuple. Applying Lemma 4.1 we conclude that in this case $\mathbf{A}$ is abelian.

E Proofs of Corollary 5.6

Proposition E.1. *Let t be a p-ary cyclic term operation of an algebra $\mathbf{A}$.*

(a) *If $\mathbf{A} = (\{0,1\}; \vee)$, then $t(x_1, \ldots, x_p) = \bigvee_{i=1}^{p} x_i$.*

(b) *If $\mathbf{A} = (\{0,1\}; \mathrm{maj})$ and $\mathbf{c} \in A^p$ is such that $c_i = a$ for $i \leq k$ and $c_i = b$ else, then $t(\mathbf{c}) = a$ if $k > p/2$ (we necessarily have $p > 2$). In particular*

$$t(\underbrace{x, x, \ldots, x}_{k\times}, \underbrace{y, y, \ldots, y}_{l\times}, \underbrace{z, z, \ldots, z}_{m\times})$$

is the ternary majority operation on $\{0,1\}$ whenever $k + l > m$, $k + m > l$, and $l + m > k$.

(c) *If $\mathbf{A}$ is a simple affine Mal'cev algebra of size q, then*

$$t(\underbrace{x, x, \ldots, x}_{k\times}, \underbrace{y, y, \ldots, y}_{l\times}, \underbrace{z, z, \ldots, z}_{m\times})$$

is the Mal'cev operation $x - y + z$ whenever $p^{-1}k = p^{-1}m = 1 \pmod q$ and $p^{-1}l = -1 \pmod q$.

Proof. The part (a) is clear. For (b) note that if we prove the first part of (b), then the ternary operation is the majority by cyclic shifts of arguments. Let $\mathbf{c}$ and $f(x,y,z) = t(x, \ldots, x, y, \ldots, y, z \ldots, z)$ be as in the statement. Note that, by cyclic shifts of arguments $f(x,x,y) = f(y,x,x)$, i.e., f is either the majority operation or the projection onto the 2nd coordinate. In both cases $t(\mathbf{c}) = a$, as required. For (c) note that a cyclic operation in a simple affine Mal'cev algebra is equal to $\sum_{i=1}^{p} ax_i \pmod q$ where $pa = 1 \pmod q$. By simple arithmetic we conclude that if the conditions on k, l, m hold, then the ternary operation is $x - y + z$. □

Corollary 5.6. *Every minimal Taylor algebra* $\mathbf{A}$ *has a ternary term operation* f *such that if* (a,b) *is an edge witnessed by* θ *on* $\mathbf{E} = \mathrm{Sg}_{\mathbf{A}}(a,b)$*, then*

- *if* (a,b) *is a semilattice edge, then* $f(x,y,z) = x \vee y \vee z$ *on* $\{a,b\}$ *(where* b *is the top);*
- *if* (a,b) *is a majority edge, then* f *is the majority operation on* $\mathbf{E}/\theta$ *(which has two elements);*
- *if* (a,b) *is an abelian edge, then* $f(x,y,z) = x - y + z$ *on* $\mathbf{E}/\theta$*.*

Proof. Choose positive integers p, k, l such that $p = 1 \pmod{|A|!}$, $k = 1 \pmod{|A|!}$, $2k + l = p$, and $2k > l$. Note that $\mathbf{A}$ has a cyclic operation t of arity p: every prime divisor of p is greater than $|A|$ and such a term p can be obtained by a star composition of terms for these primes. Define $f(x,y,z)$ by $t(\underbrace{x,x,\dots,x}_{k\times},\underbrace{y,y,\dots,y}_{l\times},\underbrace{z,z,\dots,z}_{k\times})$ and the conclusion follows from Proposition E.1. □

F Proof of Theorem 7.1

Lemma F.1. *Suppose* $\mathbf{A}$ *is a minimal Taylor algebra,* $\emptyset \neq B \subsetneq C \subseteq A$*, and* $\mathrm{Sg}_{\mathbf{A}}(C^n \setminus B^n) \cap B^n = \varnothing$ *for every* n*. Then for every* $f \in \mathrm{Clo}_n(\mathbf{A})$ *and every essential coordinate* i *of* f *we have* $f(\mathbf{a}) \notin B$ *whenever* $\mathbf{a} \in C^n$ *is such that* $a_i \in C \setminus B$*.*

Proof. Any cyclic term operation satisfies the required property (by using the compatibility with $\mathrm{Sg}_{\mathbf{A}}(C^n \setminus B^n)$ on cyclic permutations of $\mathbf{a}$) and the property is stable under identifying and permuting coordinates (and introducing dummy ones). The claim now follows from Proposition 3.5. □

Theorem 7.1. *Let* $\mathbf{A}$ *be a minimal Taylor algebra and* B *an absorbing set of* $\mathbf{A}$*. Then* B *is a subuniverse of* $\mathbf{A}$*.*

Proof. Let f be a witness for B absorbing $\mathbf{A}$ and assume, for a contradiction, that B is not a subuniverse.

Let $\mathbf{A}'$ be the algebra with universe A consisting of all operations from $\mathrm{Clo}(\mathbf{A})$ that preserve B. Since $\mathbf{A}'$ is a proper reduct of $\mathbf{A}$, it is not Taylor, so some quotient of a sublagebra of $\mathbf{A}'$ is a two-element algebra whose every operation is a projection. In other words, there exist disjoint nonempty sets B_0 and B_1 such that every operation t from $\mathrm{Clo}(\mathbf{A})$ preserving B acts like a projection on $\{B_0, B_1\}$. Note that f preserves B, therefore it has this property and we assume, without loss of generality, that f acts like the first projection on $\{B_0, B_1\}$.

It follows from the previous paragraph that, for every n, the relation $S_n = (B_0 \cup B_1)^n \setminus B_0^n$ (just like any other relation "built" over blocks B_0 and B_1) is compatible with every operation from $\mathrm{Clo}(\mathbf{A})$ preserving B and is thus pp-definable from $\mathrm{Inv}(\mathbf{A})$ and B. Let T_n be the relation defined by the same pp-definition with each conjunct $B(x)$ replaced by the void $A(x)$. Since B absorbs $\mathbf{A}$ by f, we also know that S_n absorbs T_n by the same f (see Lemma 2.9 in [4]).

Assume that $T_n \cap B_0^n \neq \emptyset$ and choose $\mathbf{a} \in T_n \cap B_0^n$ and $\mathbf{b} \in B_1^n$. Then $f(\mathbf{a},\mathbf{b},\dots,\mathbf{b})$ belongs to B_0^n because f acts like the first projection on $\{B_0, B_1\}$, and it also belongs to S_n because S_n absorbs T_n by f. This contradiction shows that $T_n \cap B_0^n = \emptyset$ for every n. Note that $S_n \subseteq T_n$ and $T_n \in \mathrm{Inv}(\mathbf{A})$, hence $\mathrm{Sg}_{\mathbf{A}}((B_0 \cup B_1)^n \setminus B_0^n) \cap B_0^n = \emptyset$ for every n. From Lemma F.1 it now follows that f cannot act like a projection on $\{B_0, B_1\}$, a contradiction. □

G Proof of Proposition 7.8

Proposition 7.8. *Let* $\mathbf{A}$ *be an algebra. The relation* $R(x,y,z) = B(x) \vee (y = z)$ *is a subuniverse of* $\mathbf{A}^3$ *if and only if for every* $f \in \mathrm{Clo}_n(\mathbf{A})$ *and every essential coordinate* i *of* f*, we have* $f(\mathbf{a}) \in B$ *whenever* $\mathbf{a} \in A^n$ *is such that* $a_i \in B$*.*

Proof. For the forward implication let f be an n-ary term operation of $\mathbf{A}$ and say, without loss of generality, that the first coordinate is essential as witnessed by tuples $(c, c_2, \dots, c_n)$ and $(c', c_2, \dots, c_n)$. Take $(b, a_2, \dots, a_n) \in B \times A^{n-1}$ and note that $R(b, c, c')$, $R(a_2, c_2, c_2), \dots, R(a_n, c_n, c_n)$. Therefore

$$R(t(b, a_2, \dots, a_n), t(c, c_2, \dots, c_n), t(c', c_2, \dots, c_n))$$

and, by the choice of $c, c', c_2, \dots, c_n$, we get $t(b, a_2, \dots, a_n) \in B$, as required.

For the reverse implication we proceed by the way of contradiction and suppose that an application of an operation f to triples from R produces a triple outside. The resulting triple does not have an element of B at the first position, therefore, by the assumption, all the input triples that *have* an element of B on the first position appear on non-essential coordinates of f. The remaining triples have the same element on the second and third positions, therefore so does the resulting triple, a contradiction. □

H Proofs of Lemma 7.17 and Proposition 7.18

Lemma 7.17. *Let* C *be a center in* $\mathbf{A}$ *and let* $b \in A \setminus C$*. Then* (b,b) is not *in the subuniverse of* $\mathbf{A}^2$ *generated by* $(\{b\} \times C) \cup (C \times C) \cup (C \times \{b\})$*.*

Proof. Let $R \leq_{sd} \mathbf{A}{\times}\mathbf{B}$ be a witness of centrality. Suppose, for a contradiction that (b,b) is generated by a term operation f, so

$$f(b, \dots, b, c_1, \dots, c_i) = b = f(c_1', \dots, c_j', b, \dots, b)$$

for some $c_1, \dots, c_i, c_1', \dots, c_j' \in C$ where $i + j$ is not less than the arity of t. Therefore $f(b{+}R, \dots, b{+}R, c_1{+}R, \dots, c_i{+}R) \subseteq b{+}R$ and, denoting $D = b{+}R$, we have $f(D, \dots, D, B, \dots, B) \subseteq D$ (with i occurrences of B on the left). Similarly, we obtain

$f(B,\dots,B,D,\dots,D)\subseteq D$ with j occurrences of B. It follows that the binary operation obtained from f by identifying the first j variables to x and the rest to y witnesses the non-trivial binary absorption $D\trianglelefteq_2 \mathbf{B}$, a contradiction with the definition of a center. □

Proposition 7.18. *Let* $\mathbf{A}$ *be an algebra,* $B\trianglelefteq\mathbf{A}$*, and*
(a) B *be a center of* $\mathbf{A}$*, or*
(b) $|B|=1$ *and* $\mathbf{A}$ *be minimal Taylor.*
Then $B\trianglelefteq_3\mathbf{A}$.

Proof. Let n be the minimal number such that $B\trianglelefteq_{n+1}\mathbf{A}$ and assume, striving for a contradiction, that $n>2$. By Lemma 7.16 there exist $\mathbf{a}^1,\dots,\mathbf{a}^n\in A^n$ such that $a^i_j\in B$ for $i\neq j$ and $\mathrm{Sg}_{\mathbf{A}^n}(\mathbf{a}^1,\dots,\mathbf{a}^n)\cap B^n\neq\emptyset$. Put $R=\mathrm{Sg}_{\mathbf{A}^n}(\mathbf{a}^1,\dots,\mathbf{a}^n)$ and assume that R is an inclusion minimal relation among all choices of $\mathbf{a}^1,\dots,\mathbf{a}^n\in A^n$.

By σ we denote the binary relation defined by

$$\mathrm{Sg}_{\mathbf{A}^2}\left((\{a^n_n\}\times B)\cup(B\times\{a^n_n\})\right).$$

and we pp-define $R'\leq\mathbf{A}^{2n-2}$ by the formula

$$R'(x_1,\dots,x_{n-1},x'_1,\dots,x'_{n-1})=\exists x_n,x'_n:\ R(x_1,\dots,x_n)\wedge R(x'_1,\dots,x'_n)\wedge\sigma(x_n,x'_n).$$

For $i\in\{1,\dots,n\}$ by $\mathbf{c}^i$ we denote $\mathbf{a}^i$ take away the last coordinate. By the definition of R' and σ, we have $\mathbf{c}^i\mathbf{c}^n,\mathbf{c}^n\mathbf{c}^i\in R'$ for every $i\in\{1,\dots,n-1\}$. Moreover, these $2n-2$ tuples satisfy the condition in the second part of Lemma 7.16. Therefore, if $R'\cap B^{2n-2}=\emptyset$, then $B\not\trianglelefteq_{2n-2}\mathbf{A}$. But $B\trianglelefteq_{n+1}\mathbf{A}$ and $2n-2\geq n+1$, a contradiction.

It now remains to show that $R'\cap B^{2n-2}=\emptyset$. Assuming the converse, there exist $\mathbf{d},\mathbf{d}'\in R\cap(B^{n-1}\times A)$ such that $(d_n,d'_n)\in\sigma$. Let $E=\mathrm{proj}_n(R\cap(B^{n-1}\times A))$. Since R was chosen inclusion minimal, we get $\mathrm{proj}_n(R)=\mathrm{Sg}_{\mathbf{A}}(B\cup\{e\})$ for every $e\in E$ (otherwise we could replace $\mathbf{a}^n$ by a tuple $\mathbf{b}\in R\cap(B^{n-1}\times\{e\})$). Let $E'=E+\sigma$. Since $B\cup\{d_n,d'_n\}\subseteq E'$, we have $a^n_n\in E'$. Hence, for $E''=a^n_n+\sigma$ we have $B\subseteq E''$ and $E''\cap E\neq\emptyset$. Therefore, $a^n_n\in E''$ and $(a^n_n,a^n_n)\in\sigma$.

This cannot happen given assumption (a) because of Lemma 7.17. In case (b), that is, $B=\{b\}$, we would get (by Proposition 3.3) that $\{b,a^n_n\}$ is a subuniverse of $\mathbf{A}$ and therefore (b,a^n_n) is a semilattice edge, which contradicts Lemma 7.11. □

I Proof of Proposition 7.22

Lemma I.1. *Suppose* $(a,a,\dots,a)\notin\mathrm{Sg}_{\mathbf{A}}\{a,b\}^n\setminus\{a\}^n$ *for every* n*. Then there exist subuniverses* B *and* C *such that* $B\subsetneq C$*,* $a\in C\setminus B$*,* $b\in B$*, and* $C^n\setminus(C\setminus B)^n\in\mathrm{Inv}(\mathbf{A})$ *for every* n.

Proof. Let $C=\mathrm{Sg}_{\mathbf{A}}(\{a,b\})$ and $B\subsetneq C$ be a maximal subuniverse of $\mathbf{A}$ such that $(a,a,\dots,a)\notin\mathrm{Sg}_{\mathbf{A}}(C^n\setminus(C\setminus B)^n)$ for every n. Since B can be chosen equal to $\{b\}$, such B exists.

Put $S_n=\mathrm{Sg}_{\mathbf{A}}\,C^n\setminus(C\setminus B)^n$, let us show that $S_n=C^n\setminus(C\setminus B)^n$ for every n. Assume the opposite. Then there exists $(a_1,\dots,a_n)\in S_n\cap(C\setminus B)^n$. Since the algebra is idempotent, $\{a_1\}\times\dots\times\{a_n\}\times C^s\subseteq S_{n+s}$ for every $s\geq0$. Let $m\in\{0,1,\dots,n-1\}$ be the maximal number such that $\{a_1\}\times\dots\times\{a_m\}\times C^s\nsubseteq S_{m+s}$ for every $s\geq0$. Then for some s' we have $\{a_1\}\times\dots\times\{a_{m+1}\}\times C^{s'}\subseteq S_{m+s'+1}$. Since the algebra is idempotent, $\{a_1\}\times\dots\times\{a_{m+1}\}\times C^s\subseteq S_{m+s+1}$ for every $s\geq s'$. Put

$$R_{s+1}(x_1,\dots,x_{s+1})=S_{m+s+1}(a_1,\dots,a_m,x_1,\dots,x_{s+1}).$$

By the definition of m we know that $R_{s+1}\neq C^{s+1}$. Since $\mathbf{A}$ is idempotent and R_{s+1} contains all tuples with b, we have $(a,a,\dots,a)\notin R_{s+1}$. Since S_{m+s+1} is symmetric, $C^{s+1}\setminus(C\setminus\{a_{m+1}\})^{s+1}\subseteq R_{s+1}$. Put $B'=B\cup\{a_{m+1}\}$. By the definition we have $R_{s+1}\in\mathrm{Inv}(\mathbf{A})$ and $C^{s+1}\setminus(C\setminus B')^{s+1}\subseteq R_{s+1}$. Therefore, $(a,a,\dots,a)\notin\mathrm{Sg}_{\mathbf{A}}(C^n\setminus(C\setminus B')^n)\subseteq R_n$ for every $n\geq s'+1$. Since $\mathbf{A}$ is idempotent, $(a,a,\dots,a)\notin\mathrm{Sg}_{\mathbf{A}}(C^n\setminus(C\setminus B')^n)$ for every $n\geq1$, which contradicts our assumption about the maximality of B. □

Lemma I.2. *Suppose* $\{b\}$ *is not an absorbing subuniverse of* $\mathbf{A}$ *and there doesn't exist a weak abelian edge coming from* b*. Then there exist subuniverses* C *and* D *such that* $\emptyset\neq C\subsetneq D$*,* $b\in D\setminus C$*, and* $C\trianglelefteq_2\mathbf{D}$.

Proof. By Lemma 7.16 for every n there exists a n-ary relation $R\in\mathrm{Inv}(\mathbf{A})$ such that $(b,b,\dots,b)\notin R$ but for every i there exists tuple from R having b at all coordinates but i-th. Choose an inclusion maximal relation $R'\supseteq R$ from $\mathrm{Inv}(\mathbf{A})$ such that $(b,b,\dots,b)\notin R'$. By Lemma 2.4 from [45] R' is a key relation and $(b,b,\dots,b)$ is a key tuple for R'. Consider the pattern of the relation R' (see [45] for the definition). By Theorem 3.1 in [45] the pattern is an equivalence relation having at most one class containing more than one element (coordinate). Assume that this large class has more than 2 elements (coordinates). Substituting b for other coordinates of R' we get a relation of arity at least 3 with full pattern. Then by Theorem 3.11 in [45] there exists a subuniverse $C\leq\mathbf{A}$ containing b and a congruence on $\mathbf{C}$ such that $\mathbf{C}/\sigma$ is an affine algebra. Thus, we get an abelian edge coming from b.

Assume that the large class has exactly 2 elements (coordinates). Substituting b for these two coordinates to R' we get a relation with trivial pattern. Thus, in all other cases for every n we can get a key relation $S\in\mathrm{Inv}(\mathbf{A})$ of arity n with trivial pattern with key tuple $(b,b,\dots,b)$. By Theorem 3.2 in [45], there exist $b_1,\dots,b_n$ such that

$$(\{b,b_1\}\times\dots\times\{b,b_n\})\setminus\{b\}^n\subseteq S.$$

Since n can be chosen arbitrary large and we can substitute the constant b to S, there exists $c\in A$ such that for every n

$$(b,b,\dots,b)\notin\mathrm{Sg}_{\mathbf{A}}((\{b,c\}^n)\setminus\{b\}^n).$$

It remains to apply Lemma I.1 and Theorem 7.6((b) implies (a)). □

Lemma I.3. *Suppose* $\mathbf{C}\trianglelefteq_2\mathbf{D}\leq\mathbf{A}$ *such that* $\emptyset\neq C\subsetneq D$*,* $b\in D\setminus C$*. Then there exists* $c\in C$ *such that* (b,c) *is a semilattice edge.*

Proof. Choose $c \in C$ such that the set $\mathrm{Sg}_{\mathbf{A}}(b, c)$ is inclusion minimal. Put $\sigma = \mathrm{Sg}_{\mathbf{A}^2}\{(b, c), (c, b)\}$.

Let us show that $(c, c) \in \sigma$. Since $C \trianglelefteq_2 \mathbf{D}$, applying a binary absorbing term operation to tuples $(b, c), (c, b)$ we obtain a tuple $(c_1, c_2) \in \sigma \cap C^2$. Then for $C' = C + \sigma$ we have $b, c_1, c_2 \in C'$. By the minimality of $\mathrm{Sg}_{\mathbf{A}}(b, c)$ we have $c \in C'$, hence $(c, c_3) \in \sigma$ for some $c_3 \in C$. Then for $C'' = c + \sigma$ we have $b, c_3 \in C''$, which, by the minimality of $\mathrm{Sg}_{\mathbf{A}}(b, c)$, gives $c \in C''$, hence $(c, c) \in \sigma$.

Therefore, there exists a term operation t such that $t(b, c) = t(c, b) = c$. By Proposition 3.3 $\{b, c\}$ is a subuniverse of $\mathbf{A}$ and therefore (b, c) is a semilattice edge. □

Proposition 7.22. *Let* $\mathbf{A}$ *be a minimal Taylor algebra and* $b \in B$*. If there are no semilattice or weak abelian edges coming from* b *then* $\{b\} \trianglelefteq \mathbf{A}$.

Proof. It follows from Lemma I.2 and Lemma I.3. □

J Proofs from Section 8

Theorem 8.1. *The following are equivalent for any minimal Taylor algebra* $\mathbf{A}$.

(a) $\mathbf{A}$ *has bounded width.*
(b) $\mathbf{A}$ *is* a*-free.*

Proof. Combining the results of [6, 19] and [16], $\mathbf{A}$ has bounded width if and only if there does not exist a homomorphic image of a subalgebra of $\mathbf{A}$ that is an abelian algebra. To show that (a) implies (b), it suffices to observe that if $\mathbf{A}$ contains an abelian edge (a, b) and congruence θ witnesses that, then the algebra $\mathrm{Sg}_{\mathbf{A}}(a, b)/\theta$ is abelian. Finally, to show that (b) implies (a), assume that there is a subalgebra $\mathbf{B}$ of $\mathbf{A}$ and its congruence θ such that $\mathbf{B}/\theta$ is abelian. Then for any a, b from different blocks of θ, the pair (a, b) is a weak abelian edge. Choosing maximal congruence above θ, say θ', and appropriate elements a', b' which are θ' related to a, b respectively we find an abelian edge. □

Theorem 8.2. *The following are equivalent for any minimal Taylor algebra* $\mathbf{A}$.

(a) $\mathbf{A}$ *has few subpowers.*
(b) $\mathbf{A}$ *is* s*-free.*
(c) *No subalgebra of* $\mathbf{A}$ *has a nontrivial 2-absorbing subuniverse.*

Proof. First, we show that (a) implies (b). If $\mathbf{A}$ contains a semilattice edge, then by Lemma 5.3 it also has a subalgebra term equivalent to a 2-element semilattice. It is known from [12, 34] that a semilattice does not have few subpowers.

Next, to show that (b) implies (c) let $B \subseteq \mathbf{A}$ be a 2-absorbing subuniverse. Then by Lemma 7.3 the set B is asm-closed. Since abelian and majority edges are not directed, this implies that $B = \mathbf{A}$.

Finally, we prove that (c) implies (a) we observe that (c) is the *HBAF* condition from [9]. Theorem 1.4 from the same paper claims that this condition implies that $\mathbf{A}$ has a *cube* term, which is equivalent to having few subpowers by [12]. □

Theorem 8.3. *The following are equivalent for any minimal Taylor algebra* $\mathbf{A}$.

(b) $\mathbf{A}$ *is* m*-free.*
(c) *Every subalgebra of* $\mathbf{A}$ *has a unique 3-minimal absorbing subuniverse.*
(d) *If* $\mathbf{C} \trianglelefteq_3 \mathbf{B} \leq \mathbf{A}$ *then* $\mathbf{C} \trianglelefteq_2 \mathbf{B}$.

Proof. By Proposition 7.13 (b) implies (d). By Corollary 7.10 (d) implies (c). The fact that every majority edge defines two disjoint 3-absorbing subuniverses shows that (c) implies (b). □

Theorem 8.4. *The following are equivalent for any minimal Taylor algebra* $\mathbf{A}$.

(a) $\mathbf{A}$ *has a Mal'cev term operation.*
(b) $\mathbf{A}$ *is* sm*-free.*
(c) *No subalgebra of* $\mathbf{A}$ *has a nontrivial absorbing subuniverse.*
(d) *No subalgebra of* $\mathbf{A}$ *has a nontrivial center.*

Proof. (a) $\Longrightarrow$ (b). If $\mathbf{A}$ has a semilattice or majority edge (a, b) witnessed by a congruence θ, then the algebra $\mathrm{Sg}_{\mathbf{A}}(a, b)/\theta$ has a Mal'cev term operation, which is impossible by Proposition 5.2.

(b) $\Longrightarrow$ (c). By Lemma 7.11 every absorbing set is as-closed. However, since the graph of $\mathbf{A}$ is a-connected, such a set has to be the whole algebra.

(c) $\Longrightarrow$ (a). Condition (c) is the property *HAF* from [9]. By Theorem 1.4 from the same paper $\mathbf{A}$ has a Mal'cev term operation.

(c) $\Longrightarrow$ (d) follows from the theorem on absorption,

(d) $\Longrightarrow$ (b) follows from Lemma 7.15. □

Theorem 8.5. *The following are equivalent for any minimal Taylor algebra* $\mathbf{A}$.

(a) $\mathbf{A}$ *has a* wnu operations *i.e. an operation satisfying* $f(y, x, \dots, x) = f(x, y, x, \dots, x) = \cdots = f(x, \dots, x, y)$ *of every arity greater than or equal to 2.*
(b) $\mathbf{A}$ *is* am*-free.*

Proof. By Proposition 5.2 every edge can only have one type of Taylor operation. Abelian edge cannot have weak near-unanimity term operations (WNU) of all arities and majority edge cannot have a commutative binary operation. Thus, (a) implies (b).

Let us show that (b) implies (a). By Theorem 8.1, $\mathbf{A}$ has bounded width, and by Theorem 6.1 from [28] the identities defining a WNU of arity $n \geq 3$ are realized in any algebra with bounded width, which shows the existence of WNUs of all arities greater than 2. It remains to obtain a WNU of arity 2, i.e. a binary commutative operation.

It is enough to prove that for any a, b there is a term $f_{ab}(x, y)$ such that $f_{ab}(a, b) = f_{ab}(b, a)$. Indeed, suppose such terms exist. Then we can use the argument from the proof of Proposition 7.13, to obtain a binary commutative operation. The argument goes as follows. We gradually build a term operation working for all pairs (a, b), (c, d), … as follows

$$f_{f_{ab}(c,d) f_{ab}(d,c)}(f_{ab}(x, y), f_{ab}(y, x))$$

to eventually obtain a commutative binary operation.

Suppose we do not have such terms, and let $\mathbf{A}$ be the smallest so that the claim fails. Take $a, b \in A$ so that for every t we have $t(a, b) \neq t(b, a)$; $\mathrm{Sg}_{\mathbf{A}}(a, b) = A$ as otherwise we have a contradiction with the minimality of $\mathbf{A}$.

The first step is to prove that there is a $\mathbf{B} \trianglelefteq_2 \mathbf{A}$. Indeed, take α a maximal congruence on $\mathbf{A}$ let $\mathbf{C} = \mathbf{A}/\alpha$ and $c = a/\alpha$, $d = b/\alpha$. The relation $\mathrm{Sg}_{\mathbf{C}^6}(ccdddc, cdcdcd, dcccdd)$ cannot be, after dropping redundant coordinates following the pattern from Section D.1, full as then (a, b) would be a weak majority edge, and inside it we would find a majority edge. Since $\mathbf{A}$ has bounded width, $\mathbf{C}$ cannot be abelian. By Theorem 4.2 we obtain an irredundant, proper subalgebra of $\mathbf{C}^2$ and by Proposition 4.3 we obtain a central relation. In the case when $\mathbf{C}$ has no 2-absorbing subuniverse we use Proposition 7.18 and Proposition 7.13 to obtain a 2-absorbing subuniverse, which is a contradiction.

We showed that $\mathbf{C}$ has a proper 2-absorbing subuniverse and thus so does $\mathbf{A}$, let $\mathbf{B} \trianglelefteq_2 \mathbf{A}$. The final step is to get $f(x, y)$ such that $f(a, b), f(b, a) \in \mathbf{B}$. If we can accomplish that then, by minimality of $\mathbf{A}$ we let $t'(x, y)$ be such that $t'(f(a, b), f(b, a)) = t'(f(b, a), f(a, b))$ and then $t(x, y) = t'(f(x, y), f(y, x))$ satisfies $t(a, b) = t(b, a)$ which is a contradiction.

To that end we will show that $A = B \cup \{a, b\}$ and that the only evaluations of any cyclic term, that produce elements outside of B are the idempotent evaluations. Take cyclic term $c(x_1, \dots, x_p)$ and let $g(x, y)$ be such that $g(a, b) \in B$; circle-compose c with $g(\cdots g(g(x_1, x_2), x_3), \dots, x_p)$ to obtain a cyclic term c' of arity p. Note that all, except the idempotent, evaluations of c' are in B. This confirms the structure of $\mathbf{A}$. Any binary reduct of c' can be taken for f and we are done. ◻

Theorem 8.6. *The following are equivalent for any minimal Taylor algebra* $\mathbf{A}$.

- (a) $\mathbf{A}$ *has a majority term operation.*
- (a') $\mathbf{A}$ *has a near unanimity term operation.*
- (b) $\mathbf{A}$ *is* as*-free.*

Proof. Clearly (a) implies (a'). Since both a two-element semilattice and an affine Mal'cev algebra do not have a near unanimity term operation, by Theorem 5.1 (a') implies (b).

Let us show that (b) implies (a). If $\mathbf{A}$ has a weak abelian edge then we can choose appropriate elements to get an abelian edge. Hence, by Proposition 7.22 every 1-element set $\{b\}$ is a center and the relation $R(x, y) = (x = b) \vee (y = b)$ is a subuniverse of $\mathbf{A}^2$. Consider any p-ary cyclic term t with $p = 2k + 1$. Since t preserves R we obtain $t(b, b, \dots, b, a_1, \dots, a_k) = b$ for any $a_1, \dots, a_k \in A$. Hence

$$f(x, y, z) = t(\underbrace{x, x, \dots, x}_{k}, \underbrace{y, y, \dots, y}_{k}, z)$$

is a majority term operation. ◻