Sensitive instances of the Constraint Satisfaction Problem

Libor Barto
Department of Algebra, Faculty of Mathematics and Physics, Charles University, Sokolovska 83, 186 00 Praha 8, Czech Republic
barto@karlin.mff.cuni.cz

Marcin Kozik
Jagiellonian University, Poland
marcin.kozik@uj.edu.pl

Johnson Tan
Department of Mathematics, University of Illinois, Urbana-Champaign, Urbana, IL USA, 61801
jgtan2@illinois.edu

Matt Valeriote
Department of Mathematics and Statistics, McMaster University, 1280 Main Street West, Hamilton, Ontario, Canada L8S 4K1
matt@math.mcmaster.ca

Abstract

We investigate the impact of modifying the constraining relations of a Constraint Satisfaction Problem (CSP) instance, with a fixed template, on the set of solutions of the instance. More precisely we investigate sensitive instances: an instance of the CSP is called sensitive, if removing any tuple from any constraining relation invalidates some solution of the instance. Equivalently, one could require that every tuple from any one of its constraints extends to a solution of the instance.

Clearly, any non-trivial template has instances which are not sensitive. Therefore we follow the direction proposed (in the context of strict width) by Feder and Vardi in [12] and require that only the locally consistent instances be sensitive. We provide a full algebraic characterization of templates with this property, under the mild assumption that they are idempotent: we show that an idempotent algebra $\mathbf{A}$ has a $k+2$ variable near unanimity term operation if and only if any instance resulting from running the $(k, k+1)$-consistency algorithm on an instance over $\mathbf{A}^2$ is sensitive.

A version of our result, without idempotency but with the sensitivity condition holding in a variety of algebras, settles a question posed by G. Bergman about systems of projections of algebras that arise from some subalgebra of a finite product of algebras.

Our results hold for infinite (albeit in the case of $\mathbf{A}^2$ idempotent) algebras as well and exhibit a surprising similarity to the strict width k condition proposed by Feder and Vardi. Both conditions can be characterized by the existence of a near unanimity operation, but the arities of the operations differ by 1.

2012 ACM Subject Classification Theory of computation → Problems, reductions and completeness; Theory of computation → Complexity theory and logic; Theory of computation → Constraint and logic programming

Keywords and phrases Constraint satisfaction problem, bounded width, local consistency, near unanimity operation, loop lemma

Funding *Libor Barto*: Research partially supported by the European Research Council (ERC) under the European Unions Horizon 2020 research and innovation programme (grant agreement No 771005), and by the Czech Science Foundation grant 18-20123S
Marcin Kozik: Research partially supported by the National Science Centre Poland grants 2014/13/B/ST6/01812 and 2014/14/A/ST6/00138
Matt Valeriote: Research partially supported by the Natural Sciences and Engineering Research Council of Canada.

1 Introduction

One important algorithmic approach to deciding if a given instance of the Constraint Satisfaction Problem (CSP) has a solution is to first consider whether it has a consistent set of local solutions. Clearly, the absence of local solutions will rule out having any (global) solutions, but in general having local solutions does not guarantee the presence of a solution. A major thrust of the recent research on the CSP has focussed on coming up with suitable notions of local consistency and then characterizing those CSPs for which local consistency implies outright consistency or some stronger property.

In this paper we will consider a new notion of local consistency and provide an algebraic characterization of it over collections of CSP instances whose constraint relations are confined to a set prescribed by a finite relational structure (sometimes called a template), an algebra (possibly infinite), or a collection of algebras. A good source for background material is the survey article [7].

Early results of Feder and Vardi [12] and also Jeavons, Cooper, and Cohen [15] establish that when a template $\mathbb{A}$ has a special type of polymorphism, called a near unanimity operation, then not only will an instance of the CSP over $\mathbb{A}$ that has a suitably consistent set of local solutions have a solution, but that any partial solution of it can always be extended to a solution. The notion of local consistency that we investigate in this paper is related to that considered by these researchers but that, as we shall see, is weaker.

Central to our investigation are *near unanimity operations*. These are operations $n(x_1, \ldots, x_{k+1})$ on a set A of arity $k+1$, for some $k > 1$, that satisfy the equalities

$$n(b, a, a, \ldots, a) = n(a, b, a, \ldots, a) = \cdots = n(a, a, \ldots, a, b) = a$$

for all a, $b \in A$. These operations have played an important role in the development of universal algebra and first appeared in the 1970's in the work of Baker and Pixley [1] and Huhn [14]. More recently they have been used in the study of the CSP [12, 15] and related questions [2, 11]. The main results of this paper can be expressed in terms of the CSP and also in algebraic terms and we start by presenting them from both perspectives. In the concluding section, Section 5, a translation of parts of our results into a relational language is provided, along with some open problems.

1.1 CSP viewpoint

In their seminal paper, Feder and Vardi [12] introduced the notion of bounded width for the class of CSP instances over a finite template $\mathbb{A}$. Their definition of bounded width was presented in terms of the logic programming language DATALOG but there is an equivalent formulation using local consistency algorithms, also given in [12]. Given a CSP instance $\mathcal{I}$ and $k < l$, the (k,l)-consistency algorithm will produce a new instance having all k variable constraints that can be inferred by considering l variables at a time of $\mathcal{I}$. This algorithm rejects $\mathcal{I}$ if it produces an empty constraint. The class of CSP instances over a finite template $\mathbb{A}$ will have width (k,l) if the (k,l)-consistency algorithm rejects all instances from the class that do not have solutions, i.e., the (k,l)-consistency algorithm can be used to decide if a given instance from the class has a solution or not. The class has bounded width if it has width (k,l) for some $k < l$.

A lot of effort, in the framework of the algebraic approach to the CSP, has gone in to analyzing various properties of instances that are the outputs of these types of local consistency algorithms. On one end of the spectrum of the research is a rather wide class of templates of bounded width [5] and on the other a very restrictive class of templates having bounded strict width [12].

To be more precise let us define a CSP *instance* $\mathcal{I}$ to be $(V, \mathcal{C})$ where V is a set of variables, and $\mathcal{C}$ is a set of constraints of the form $((x_1, \ldots, x_n), R)$ where all x_i are in V and R is an n-ary relation over (possibly infinite) sets A_i associated to each variable x_i. *A solution* of $\mathcal{I}$ is an evaluation f of variables such that, for every $((x_1, \ldots, x_n), R) \in \mathcal{C}$ we have $(f(x_1), \ldots, f(x_n)) \in R$; *a partial solution* is a partial function satisfying the same condition.

Instances produced by the (k, l)-consistency algorithm have uniformity and consistency properties that we highlight. The instance $\mathcal{I} = (V, \mathcal{C})$ is k*-uniform* if all of its constraints are k-ary and every set of k variables is constrained by a single constraint. An instance is a (k, l)*-instance* if it is k-uniform and for every choice of a set W of l variables no additional information about the constraints can be derived by restricting the instance to the variables in W. This last, very important, property can be rephrased in the following way: for every set $W \subseteq V$ of size l; every tuple in every constraint of $\mathcal{I}_{|W}$ participates in a solution to $\mathcal{I}_{|W}$ (where $\mathcal{I}_{|W}$ is obtained from $\mathcal{I}$ by removing all the variables outside of W and all the constraints that contain any such variables).

Following the algebraic approach to the CSP we replace templates $\mathbb{A}$ with algebras $\mathbf{A}$ and define CSP($\mathbf{A}$) to be the class of CSP instances whose constraint relations are amongst those relations over A that are preserved by the operations of $\mathbf{A}$ (i.e., they are subuniverses of powers of $\mathbf{A}$). A number of important questions about the CSP can be reduced to considering templates that have all of the singleton unary relations [7]; the algebraic counterpart to these types of templates are the *idempotent algebras* (all operations of the algebra satisfy $f(a, \ldots, a) = a$ for every possible argument a). As demonstrated in Example 12, several of the results in this paper do not hold in the absence of idempotency.

Consider the notion of strict width k introduced by Feder and Vardi [12, Section 6.1.2]. For $\mathbb{A}$ a template, the class of instances of the CSP over $\mathbb{A}$ has strict width k if whenever the $(k, k+1)$-consistency algorithm does not reject an instance $\mathcal{I}$ from the class then "it should be possible to obtain a solution by greedily assigning values to the variables one at a time while satisfying the inferred k-constraints." It can be seen that this is equivalent to the property that if $\mathcal{I}$ is the result of applying the $(k, k+1)$-consistency algorithm to an instance from the class that has some solution, then any partial solution of $\mathcal{I}$ can be extended to a solution. In [12, Theorem 25] Feder and Vardi prove that this is also equivalent to $\mathbb{A}$ having a near unanimity operation of arity $k+1$ as a polymorphism.

In contrast to the situation for finite templates, when considering this extension property for $(k, k+1)$-instances of CSP($\mathbf{A}$) for $\mathbf{A}$ an algebra, one cannot conclude, in general, that $\mathbf{A}$ will have a $(k+1)$-ary near unanimity term operation, even if $\mathbf{A}$ is assumed to be finite and idempotent.

▶ **Example 1.** Consider the rather trivial algebra $\mathbf{A}$ that has universe $\{0, 1\}$ and no basic operations. If $\mathcal{I}$ is a $(2, 3)$-instance over $\mathbf{A}$ then since every binary relation over $\{0, 1\}$ is invariant under the majority operation on $\{0, 1\}$ it follows that every partial solution of $\mathcal{I}$ can be extended to a solution. Of course, $\mathbf{A}$ does not have a near unanimity term operation of any arity.

What this example demonstrates is that in general, for a fixed k, the k-ary constraint relations arising from an algebra do not capture that much of the structure of the algebra. Example 12 provides further evidence for this.

Our first theorem shows that for finite idempotent algebras $\mathbf{A}$, by considering a slightly bigger set of $(k, k+1)$-instances, over CSP($\mathbf{A}^2$), rather than over CSP($\mathbf{A}$), we can detect the presence of a $(k+1)$-ary near unanimity term operation. We note that every $(k, k+1)$-instance over $\mathbf{A}$ can be easily encoded as a $(k, k+1)$-instance over $\mathbf{A}^2$.

▶ **Theorem 2.** *Let $\mathbf{A}$ be a finite, idempotent algebra and $k > 1$. The following are equivalent:*
1. *$\mathbf{A}$ (or equivalently $\mathbf{A}^2$) has a near unanimity term operation of arity $k+1$;*
2. *in every $(k, k+1)$-instance over $\mathbf{A}^2$, every partial solution extends to a solution;*
3. *in every $(k, k+1)$-instance over $\mathbf{A}^2$ **on** $k+2$ **variables**, every partial solution extends to a solution.*

When $\mathbf{A}$ is the algebra of polymorphisms of a finite template $\mathbb{A}$ that has all of the singleton unary relations, then we obtain another characterization of when the class of CSP instances over $\mathbb{A}$ has strict width k, namely that the partial solution extension property need only be checked for $(k, k+1)$-instances (over A^2) in $k+2$ variables. In Theorem 10 we extend our result to infinite idempotent algebras by working with local near unanimity term operations.

Going back the original definition of strict width: "it should be possible to obtain a solution by greedily assigning values to the variables one at a time while satisfying the inferred k-constraints" we note that the requirement that the assignment should be greedy is rather restrictive. The main theorem of this paper investigates an arguably more natural concept where the assignment need not be greedy. Formally, our condition is that in a $(k, k+1)$-instance every tuple in every constraint extends to a solution. Equivalently, every $(k, k+1)$-instance is a (k, n)-instance, where n is the number of variables present in the instance. Even more naturally: removing any tuple from any constraining relation of a $(k, k+1)$-instance alters the space of solutions of that instance — we call such instances *sensitive*. We provide the following characterization.

▶ **Theorem 3.** *Let $\mathbf{A}$ be a finite, idempotent algebra and $k > 1$. The following are equivalent:*
1. *$\mathbf{A}$ (or equivalently $\mathbf{A}^2$) has a near unanimity term operation of arity $k+2$;*
2. *every $(k, k+1)$-instance over $\mathbf{A}^2$ is sensitive;*
3. *every $(k, k+1)$-instance over $\mathbf{A}^2$ **on** $k+2$ **variables** is sensitive.*

Exactly as in Theorem 2 we can consider infinite algebras at the cost of using local near unanimity term operations (see Theorem 11).

In conclusion we investigate a natural property of instances motivated by the definition of strict width and provide a characterization of this new condition in algebraic terms. A surprising conclusion is that the new concept is, in fact, very close to the strict width concept, i.e., for a fixed k one characterization is equivalent to a near unanimity operation of arity $k+1$ and the second of arity $k+2$.

1.2 Algebraic viewpoint

Our work has as an antecedent the papers of Baker and Pixley [1] and of Bergman [8] on algebras having near unanimity term operations. In these papers the authors considered subalgebras of products of algebras and systems of projections associated with them. Baker and Pixley showed that in the presence of a near unanimity term operation, such a subalgebra is closely tied with its projections onto small sets of coordinates.

A *variety of algebras* is a class of algebras of the same signature that is closed under taking homomorphic images, subalgebras, and direct products. For $\mathbf{A}$ an algebra, $\mathcal{V}(\mathbf{A})$ denotes the smallest variety that contains $\mathbf{A}$ and is called the *variety generated by* $\mathbf{A}$. A variety $\mathcal{V}$ has a near unanimity term of arity $k+1$ if there is some $(k+1)$-ary term of $\mathcal{V}$ whose interpretation in each member of $\mathcal{V}$ is a near unanimity operation.

Here is one version of the Baker-Pixley Theorem:

▶ **Theorem 4** (see Theorem 2.1 from [1]). *Let* $\mathbf{A}$ *be an algebra and* $k > 1$*. The following are equivalent:*

1. $\mathbf{A}$ *has a* $(k+1)$*-ary near unanimity term operation;*
2. *for every* $r > k$ *and every* $\mathbf{A}_i \in \mathcal{V}(\mathbf{A})$*,* $1 \leq i \leq r$*, every subalgebra* $\mathbf{R}$ *of* $\prod_{i=1}^{r} \mathbf{A}_i$ *is* ***uniquely*** *determined by the projections of* R *on all products* $A_{i_1} \times \cdots \times A_{i_k}$ *for* $1 \leq i_1 < i_2 < \cdots < i_k \leq r$*;*
3. *the same as condition 2, with* r *set to* $k+1$*.*

In other words, an algebra has a $(k+1)$-ary near unanimity term operation if and only if every product of algebras from $\mathcal{V}(\mathbf{A})$ is uniquely determined by its system of k-fold projections into its factor algebras. A natural question, extending the result above, was investigated by Bergman [8]: when does a "system of k-fold projections" arise from a product algebra?

Note that such a system can be viewed as a k-uniform CSP instance: indeed, following the notation of Theorem 4, we can introduce a variable x_i for each $i \leq r$ and a constraint $((x_{i_1}, \ldots, x_{i_k}); \mathrm{proj}_{i_1,\ldots,i_k} R)$ for each $1 \leq i_1 < i_2 < \cdots < i_k \leq r$. In this way the original relation R consists of solutions of the created instance (but in general will not contain all of them). Note that, in this particular instance, different variables can be evaluated in different algebras. We will say that $\mathcal{I}$ is a CSP instance *in the variety* $\mathcal{V}$ *(denoted* $\mathcal{I} \in \mathsf{CSP}(\mathcal{V})$*)* if all the constraining relations of $\mathcal{I}$ are algebras in $\mathcal{V}$. In the language of the CSP, Bergman proved the following:

▶ **Theorem 5** ([8]). *If* $\mathcal{V}$ *is a variety that has a* $(k+1)$*-ary near unanimity term then every* $(k, k+1)$*-instance in* $\mathcal{V}$ *is sensitive.*

In commentary that Bergman provided on his proof of this theorem he noted that a stronger conclusion could be drawn from it and he proved the following theorem. We note that this theorem anticipates the results from [12] and [15] dealing with templates having near unanimity operations as polymorphisms.

▶ **Theorem 6** ([8]). *Let* $k > 1$ *and* $\mathcal{V}$ *be a variety. The following are equivalent:*

1. $\mathcal{V}$ *has a* $(k+1)$*-ary near unanimity term;*
2. *any partial solution of a* $(k, k+1)$*-instance over* $\mathcal{V}$ *extends to a solution.*

In Appendix A we present a proof of this theorem.

Theorem 5 provides a partial answer to the question that Bergman posed in [8], namely that in the presence of a $(k+1)$-ary near unanimity term, a necessary and sufficient condition for a k-fold system of algebras to arise from a product algebra is that the associated CSP instance is a $(k, k+1)$-instance.

In [8] Bergman asked whether the converse to Theorem 5 holds, namely, that if the stated equivalence holds for all k-uniform instances defined over algebras from a variety, must the variety have a $(k+1)$-ary near unanimity term? He provided examples that suggested that the answer is no, and we confirm this by proving that the condition is actually equivalent to the variety having a near unanimity term of arity $k+2$. The main result of this paper, viewed from the algebraic perspective (but stated in terms of the CSP), is the following:

▶ **Theorem 7.** *Let* $k > 1$*. A variety* $\mathcal{V}$ *has a* $(k+2)$*-ary near unanimity term if and only if each* $(k, k+1)$*-instance of the* CSP *over algebras from* $\mathcal{V}$ *is sensitive.*

The "if direction" of this theorem is proved in Section 3, while a proof of the "only if direction" is provided in Appendix D. We note that a novel and significant feature of this result is that it does not assume any finiteness or idempotency of the algebras involved.

1.3 Structure of the paper

The paper is structured as follows. In the next section we introduce local near unanimity operations and state Theorem 2 and Theorem 3 in their full power. In Section 3 we collect the proofs that establish the existence of (local) near unanimity operations. In Section 4 we provide a sketch of the proof showing that, in the presence of a near unanimity operation of arity $k+2$, the $(k, k+1)$-instances are sensitive. A full proof of this fact, which is the main contribution of this paper, can be found in Appendix D. Finally, Section 5 contains conclusions.

Appendix A and Appendix B are provided for the convenience of the reader. They prove facts required for the classification, but known before, and facts which can be proved by minor adaptations of known reasoning. Appendix C contains a proof of a new loop lemma, which can be of independent interest, and is necessary in the proof in Appendix D. Finally Appendix D contains, as already mentioned, the main technical contribution of the paper.

2 Details of the CSP viewpoint

In order to state our results in their full strength, we need to define local near unanimity operations. This special concept of local near unanimity operations is required, when considering infinite algebras.

▶ **Definition 8.** *Let $k > 1$. An algebra $\mathbf{A}$ has* local near unanimity term operations of arity $k+1$ *if for every finite subset S of A there is some $(k+1)$-ary term operation n_S of $\mathbf{A}$ such that*

$$n_S(b, a, \dots, a, a) = n_S(a, b, a, \dots, a) = \dots = n_S(a, a, \dots, b, a) = n_S(a, a, \dots, a, b) = a.$$

for all a, $b \in S$.

It should be clear that, for finite algebras, having local near unanimity term operations of arity $k+1$ and having a near unanimity term operation of arity $k+1$ are equivalent, but for arbitrary algebras they are not. The following provides a characterization of when an idempotent algebra has local near unanimity term operations of some given arity; it will be used in the proofs of Theorems 10 and 11. It is similar to Theorem 4 and is proved in Appendix A.

▶ **Theorem 9.** *Let $\mathbf{A}$ be an idempotent algebra and $k > 1$. The following are equivalent:*

1. *$\mathbf{A}$ has local near unanimity term operations of arity $k+1$;*
2. *for every $r > k$, every subalgebra of $\mathbf{A}^r$ is uniquely determined by its projections onto all k-element subsets of coordinates;*
3. *every $(k+1)$-generated subalgebra of $\mathbf{A}^{k+1}$ is uniquely determined by its projections onto all k-element subsets of coordinates.*

We are ready to state Theorem 2 in its full strength:

▶ **Theorem 10.** *Let $\mathbf{A}$ be an idempotent algebra and $k > 1$. The following are equivalent:*

1. *$\mathbf{A}$ (or equivalently $\mathbf{A}^2$) has local near unanimity term operations of arity $k+1$;*
2. *in every $(k, k+1)$-instance over $\mathbf{A}^2$, every partial solution extends to a solution;*
3. *in every $(k, k+1)$-instance over $\mathbf{A}^2$* ***on*** *$k+2$* ***variables****, every partial solution extends to a solution.*

Proof. Obviously condition 2 implies condition 3. A proof of condition 3 implying condition 1 can be found in Section 3. The implication from 1 to 2 is covered by Theorem 6. ◀

Analogously, the main result of the paper, for idempotent algebras, and the full version of Theorem 3 states:

▶ **Theorem 11.** *Let $\mathbf{A}$ be an idempotent algebra and $k > 1$. The following are equivalent:*
1. *$\mathbf{A}$ (or equivalently $\mathbf{A}^2$) has local near unanimity term operations of arity $k+2$;*
2. *every $(k, k+1)$-instance over $\mathbf{A}^2$ is sensitive;*
3. *every $(k, k+1)$-instance over $\mathbf{A}^2$* ***on*** *$k+2$* ***variables*** *is sensitive.*

Proof. Obviously condition 2 implies condition 3. For condition 3 implying condition 1 see Section 3, while for the remaining implication, see Appendix D. ◀

▶ **Example 12.** The following examples show that in Theorems 9, 10, and 11 the assumption of idempotency is necessary. For $n > 2$, let $\mathbf{S}_n$ be the algebra with domain $[n] = \{1, 2, \ldots, n\}$ and with basic operations consisting of all unary operations on $[n]$ and all non-surjective operations on $[n]$ of arbitrary arity. The collection of such operations forms a finitely generated clone, called the Słupecki clone. Relevant details of these algebras can be found in [16, Example 4.6] and [20]. It can be shown that for $m < n$, the subuniverses of $\mathbf{S}_n^m$ consist of all m-ary relations R_θ over $[n]$ determined by a partition θ of $[m]$ by

$$R_\theta = \{(a_1, \ldots, a_m) \mid a_i = a_j \text{ whenever } (i, j) \in \theta\}.$$

These rather simple relations are preserved by any operation on $[n]$, in particular by any majority operation or more generally, by any near unanimity operation.

It follows from Theorem 6 that if $k > 1$ and $\mathcal{I}$ is a $(k, k+1)$-instance of $\mathsf{CSP}(\mathbf{S}_{2k+1}^2)$ then any partial solution of $\mathcal{I}$ extends to a solution. This also implies that $\mathcal{I}$ is sensitive. Furthermore any subalgebra of $\mathbf{S}_{k+2}^{k+1}$ is determined by it projections onto all k-element sets of coordinates. As noted in [16, Example 4.6], for $n > 2$, $\mathbf{S}_n$ does not have a near unanimity term operation of any arity, since the algebra $\mathbf{S}_n^n$ has a quotient that is a 2-element essentially unary algebra.

3 Constructing near unanimity operations

In this section we collect the proofs providing, under various assumptions, near unanimity or local near unanimity operations. That is: the proofs of "3 implies 1" in Theorems 10 and Theorem 11 as well as a proof of the "if direction" from Theorem 7.

In the following proposition we construct instances over $\mathbf{A}^2$ (for some algebra $\mathbf{A}$). By a minor abuse of notation, we allow in such instances two kinds of variables: variables x evaluated in $\mathbf{A}$ and variables y evaluated in $\mathbf{A}^2$. The former kind should be formally considered as variables evaluated in $\mathbf{A}^2$ where each constraint enforces that x is sent to $\{(b, b) \mid b \in A\}$.

Moreover, dealing with k-uniform instances, we understand the condition "every set of k variables is constrained by a single constraint" flexibly: in some cases we allow for more constraints with the same set of variables, as long as the relations are proper permutations so that every constraint imposes the same restriction.

▶ **Proposition 13.** *Let $k > 1$ and let $\mathbf{A}$ be an algebra such that, for every $(k, k+1)$-instance $\mathcal{I}$ over $\mathbf{A}^2$ on $k+2$ variables every partial solution of $\mathcal{I}$ extends to a solution. Then each subalgebra of $\mathbf{A}^{k+1}$ is determined by its k-ary projections.*

Proof. Let $\mathbf{R} \leq \mathbf{A}^{k+1}$ and we will show that it is determined by the system of projections $\mathrm{proj}_I(R)$ as I ranges over all k elements subsets of coordinates. Using $\mathbf{R}$ we define the following instance $\mathcal{I}$ of $\mathsf{CSP}(\mathbf{A}^2)$. The variables of $\mathcal{I}$ will be the set $\{x_1, x_2, \ldots, x_{k+1}, y_{12}\}$ and the domain of each x_i is A, while the domain of y_{12} is A^2.

For $U \subseteq \{x_1, \ldots, x_{k+1}\}$ of size k, let C_U be the constraint with scope U and constraint relation $R_U = \mathrm{proj}_U(R)$. For U a $(k-1)$-element subset of $\{x_1, \ldots, x_{k+1}\}$, let $C_{U\cup\{y_{12}\}}$ be the constraint with scope $U \cup \{y_{12}\}$ and constraint relation $R_{U\cup\{y_{12}\}}$ that consists of all tuples $(b_v \mid v \in U \cup \{y_{12}\})$ such that there is some $(a_1, \ldots, a_{k+1}) \in R$ with $b_v = a_i$ if $v = x_i$ and with $b_{y_{12}} = (a_1, a_2)$.

The instance $\mathcal{I}$ is k-uniform and we will show that it is sensitive. Indeed every tuple in every constraining relation originates in some tuple $\mathbf{b} \in \mathbf{R}$. Setting $x_i \mapsto b_i$ and $y_{12} \mapsto (b_1, b_2)$ defines a solution that extends such a tuple.

In particular $\mathcal{I}$ is a $(k, k+1)$-instance over $\mathbf{A}^2$ with $k+2$ variables and so any partial solution of it can be extended to a solution. Let $\mathbf{b} \in A^{k+1}$ such that $\mathrm{proj}_I(\mathbf{b}) \in \mathrm{proj}_I(R)$ for all k element subsets I of $[k+1]$. Then $\mathbf{b}$ is a partial solution of $\mathcal{I}$ over the variables $\{x_1, \ldots, x_{k+1}\}$ and thus there is some extension of it to the variable y_{12} that produces a solution of $\mathcal{I}$. But there is only one consistent way to extend $\mathbf{b}$ to y_{12} namely by setting y_{12} to the value (b_1, b_2). By considering the constraint with scope $\{x_3, \ldots, x_{k+1}, y_{12}\}$ it follows that $\mathbf{b} \in R$, as required. ◀

Now we are ready to prove the first implication tackled in this section: 3 implies 1 in Theorem 10.

Proof of "3 implies 1" in Theorem 10. By Theorem 9 it suffices to show that each subalgebra of $\mathbf{A}^{k+1}$ is determined by its k-ary projections. Fortunately, Proposition 13 provides just that. ◀

We move on to proofs of "3 implies 1" in Theorem 11 and the "if" direction of Theorem 7. Similarly, as in the theorem just proved, we start with a proposition.

▶ **Proposition 14.** *Let $k > 1$ and let $\mathbf{A}$ be an algebra such that, every $(k, k+1)$-instance $\mathcal{I}$ over $\mathbf{A}^2$ on $k+2$ variables is sensitive. Then each subalgebra of $\mathbf{A}^{k+2}$ is determined by its $(k+1)$-ary projections.*

Proof. We will show that if $\mathbf{R}$ is a subalgebra of $\mathbf{A}^{k+2}$ then $\mathbf{R} = \mathbf{R}^*$ where

$$\mathbf{R}^* = \{a \in A^{k+2} \mid \mathrm{proj}_I(a) \in \mathrm{proj}_I(\mathbf{R}) \text{ whenever } |I| = k+1\}.$$

In other words, we will show that the subalgebra $\mathbf{R}$ is determined by its projections into all $(k+1)$-element sets of coordinates.

We will use $\mathbf{R}$ and $\mathbf{R}^*$ from the previous paragraph to construct a $(k, k+2)$-instance $\mathcal{I} = (V, \mathcal{C})$ with $V = \{x_5, \ldots, x_{k+2}, y_{12}, y_{34}, y_{13}, y_{24}\}$ where each x_i is evaluated in $\mathbf{A}$ while all the y's are evaluated in $\mathbf{A}^2$.

The set of constraints is more complicated. There is a *special constraint* on a *special variable set* $((y_{12}, y_{34}, x_5, \ldots, x_{k+2}), C)$ where

$$C = \{((a_1, a_2), (a_3, a_4), a_5, \ldots, a_{k+2}) \mid (a_1, \ldots, a_{k+2}) \in R^*\}.$$

The remaining constraints are defined using the relation R. For each set of variables $S = \{v_1, \ldots, v_k\} \subseteq V$ (which is different than the set for the special constraint) we define a constraint $((v_1, \ldots, v_k), D_S)$ with $(b_1, \ldots, b_k) \in D_S$ if and only if there exists a tuple $(a_1, \ldots, a_{k+2}) \in R$ such that:

- if v_i is x_j then $b_i = a_j$, and
- if v_i is y_{lm} then $b_i = (a_l, a_m)$.

Note that the instance $\mathcal{I}$ is k-uniform.

▷ Claim 15. $\mathcal{I}$ is a $(k, k+1)$-instance.

Let $S \subseteq V$ be a set of size k. If S is not the special variable set, then every tuple in the relation constraining S originates in some $(b_1, \ldots, b_{k+2}) \in R$ and, as in Proposition 13, sending $x_i \mapsto b_i$ and $y_{lm} \mapsto (b_l, b_m)$ defines a solution that extends such a tuple. We immediately conclude, that the potential failure of the $(k, k+1)$ condition must involve the special constraint.

Thus $S = \{y_{12}, y_{34}, x_5, \ldots, x_{k+2}\}$ and if $\mathbf{b}$ is a tuple from the special constraint C then there is some $(a_1, \ldots, a_{k+2}) \in R^*$ with

$$\mathbf{b} = ((a_1, a_2), (a_3, a_4), a_5, \ldots, a_{k+2}).$$

The extra variable that we want to extend the tuple $\mathbf{b}$ to is either y_{13} or y_{24}. Both cases are similar and we will only work through the details when it is y_{13}. In this case, assigning the value (a_1, a_3) to the variable y_{13} will produce an extension $\mathbf{b}'$ of $\mathbf{b}$ to a tuple over $S \cup \{y_{13}\}$ that is consistent with all constraints of $\mathcal{I}$ whose scopes are subsets of $\{y_{12}, y_{34}, x_5, \ldots, x_{k+2}, y_{13}\}$.

To see this, consider a k element subset S' of $\{y_{12}, y_{34}, x_5, \ldots, x_{k+2}, y_{13}\}$ that excludes some variable x_j. Then, by the definition of $\mathbf{R}^*$ there exists some tuple of the form $(a_1, a_2, \ldots, a_{j-1}, a'_j, a_{j+1}, \ldots, a_{k+2}) \in R$. This tuple from R can be used to witness that the restriction of $\mathbf{b}'$ to S' satisfies the constraint $D_{S'}$ since the scope of this constraint does not include the variable x_j.

Suppose that S' is a k element subset of $\{y_{12}, y_{34}, x_5, \ldots, x_{k+2}, y_{13}\}$ that excludes y_{12}. By the definition of $\mathbf{R}^*$ there is some tuple of the form $(a_1, a'_2, a_3, \ldots, a_{k+2}) \in R$. Using this tuple it follows that the restriction of $\mathbf{b}'$ to S' satisfies the constraint $D_{S'}$. This is because neither of the variables y_{12} and y_{24} are in S' and so the value $a'_2 \in A_2$ does not matter. A similar argument works when S' is assumed to exclude y_{34} and the claim is proved.

Since $\mathcal{I}$ is a $(k, k+1)$-instance over $\mathbf{A}^2$ and it has $k+2$ variables then by assumption, $\mathcal{I}$ is sensitive. We can use this to show that $R^* \subseteq R$ to complete the proof of this theorem. Let $(a_1, \ldots, a_{k+2}) \in R^*$ and consider the associated tuple $\mathbf{b} = ((a_1, a_2), (a_3, a_4), a_5, \ldots, a_{k+2}) \in C$. Since $\mathcal{I}$ is sensitive then this k-tuple can be extended to a solution $\mathbf{b}'$ of $\mathcal{I}$. Using any constraints of $\mathcal{I}$ whose scopes include combinations of y_{12} or y_{34} with y_{13} or y_{24} it follows that the value of $\mathbf{b}'$ on the variables y_{13} and y_{24} are (a_1, a_3) and (a_2, a_4) respectively. Then considering the restriction of $\mathbf{b}'$ to $S = \{x_5, \ldots, x_{k+2}, y_{13}, y_{24}\}$ it follows that $(a_1, \ldots, a_{k+2}) \in R$ since this restriction lies in the constraint relation D_S. ◀

We are in a position to provide the two final proofs in this section.

Proof of "3 implies 1" in Theorem 11. By Theorem 9 it suffices to show that each subalgebra of $\mathbf{A}^{k+2}$ is determined by its $(k+1)$-ary projections. Fortunately Propositions 14 provides just that. ◀

Proof of the "if direction" in Theorem 7. Let $\mathbf{F}$ be the free algebra in $\mathcal{V}$ generated by x and y. Let $\mathbf{R} \leq \mathbf{F}^{k+2}$ be generated by the tuples $(y, x, \ldots, x)$, $(x, y, x, \ldots, x)$, $\ldots$, $(x, \ldots, x, y)$. By Proposition 14, the algebra $\mathbf{R}$ is determined by its $(k+1)$-ary projections and so the constant tuple $(x, \ldots, x)$ belongs to $\mathbf{R}$. The term generating this tuple defines the required $(k+2)$-ary near unanimity operation. ◀

4 Consistent instances are sensitive (sketch of a proof)

In this section we provide a high-level overview of the proof, showing that if an algebra $\mathbf{A}$ (or a variety $\mathcal{V}$) has a near unanimity operation of arity $k+2$ then all the $(k, k+1)$-instances over this algebra (or the variety) are sensitive. This will prove the "only if direction" in Theorem 7 and "1 implies 2" in Theorem 11.

Let $\mathcal{I} = (V, \mathcal{C})$ be such a $(k, k+1)$-instance. On the highest level the proof is by induction on the number of variables of $\mathcal{I}$. That means that we fix a constraint $((x_1, \ldots, x_k), R)$ of $\mathcal{I}$ and a tuple $(a_1, \ldots, a_k) \in R$ and proceed to define an instance $\mathcal{J}$ over $V \setminus \{x_1, \ldots, x_k\}$. For each constraint $((y_1, \ldots, y_k), S)$ of $\mathcal{I}$ (except for $((x_1, \ldots, x_k), R)$) $\mathcal{J}$ will include the constraint $((y'_1, \ldots, y'_l), R')$ where $y'_1, \ldots, y'_l$ is an enumeration of $\{y_1, \ldots, y_k\} \setminus \{x_1, \ldots, x_k\}$ and $R' = \mathrm{proj}_{y'_1,\ldots,y'_l}\{\mathbf{b} \in R \mid b_{y_j} = a_i \text{ if } y_j = x_i\}$.

Note that the instance $\mathcal{J}$ is not k-uniform, but this problem can be easily dealt with, at least in the case when $|V| \geq 2k$. One can, for example, remove all the constraints of arity $< k$, by updating a constraining relation of some constraint, of arity k, which has bigger scope. Let's assume that $|V| > 2k + 1$ and denote the k-uniform instance obtained from $\mathcal{J}$ by $\mathcal{J}'$. In this case, at least in the finite case, our proof boils down to a reasoning which shows that inside $\mathcal{J}'$ one can find a $(k, k+1)$-instance and the conclusion then follows by induction. In the general, infinite case the $(k, k+1)$-consistency does not transfer and we need to deal with a weaker notion: we use a condition that is equivalent to the solvability of specially constructed instances called *patterns*. See appendix D for details.

The remaining case i.e., when $k + 1 < |V| \leq 2k + 1$, is different. In this case we can show directly that every $(k, k+1)$-instance compatible with $(k+2)$-ary near unanimity is, in fact, a $(k, |V|)$-instance.

Unfortunately, the full proof is fairly more complicated than the sketch indicates. In particular we need to deal with constraints of arity $< k$ and the two cases above are not separated: in order to establish even $(k, k+2)$-consistency we need to construct patterns that, only after an application of the loop lemma proved in Appendix C, provide said consistency.

5 Conclusion

We have characterized varieties that have sensitive $(k, k+1)$-instances of the CSP as those that posses a near unanimity term of arity $k+2$. From the computational perspective, the following corollary is perhaps the most interesting consequence of our results.

▶ **Corollary 16.** *Let $\mathbb{A}$ be a finite* CSP *template whose relations all have arity at most k and which has a near unanimity polymorphism of arity $k+2$. Then every instance of the* CSP *over $\mathbb{A}$, after enforcing the $(k, k+1)$-consistency, is sensitive.*

Therefore not only is the $(k, k+1)$-consistency algorithm sufficient to detect global inconsistency, we also additionally get the sensitivity property. Let us compare this result to some previous results as follows. Consider a template $\mathbb{A}$ that, for simplicity, has only unary and binary relations and that has a near unanimity polymorphism of arity $k+2 \geq 4$. Then any instance of the CSP over $\mathbb{A}$ satisfies the following.

1. After enforcing $(2, 3)$-consistency, if no contradiction is detected, then the instance has a solution [4] (this is the bounded width property).
2. After enforcing $(k, k+1)$-consistency, every partial solution on k variables extends to a solution (this is the sensitivity property).

3. After enforcing $(k+1, k+2)$-consistency, every partial solution extends to a solution [12] (this is the bounded strict width property).

For $k + 2 > 4$ there is a gap between the first and the second item. Are there natural conditions that can be placed there?

The properties of a template $\mathbb{A}$ from the first and the third item (holding for every instance) can be characterized by the existence of certain polymorphisms: a near unanimity polymorphism of arity $k + 2$ for the third item [12] and weak near unanimity polymorphisms of all arities greater than 2 for the first item [5, 17]. This paper does not give such a direct characterization for the second item (essentially, since Theorem 11 involves a square). Is there any? Moreover, there are characterizations for natural extensions of the first and the third to relational structures with higher arity relations [12, 3]. This remains open for the second item as well.

In parallel with the flurry of activity around the CSP over finite templates, there has been much work done on the CSP over infinite ω-categorical templates [9, 19]. These templates cover a much larger class of computational problems but, on the other hand, share some pleasant properties with the finite ones. In particular, the $(k, k+1)$-consistency of an instance can still be enforced in polynomial time. Corollary 16 can be extended to this setting as follows.

▶ **Corollary 17.** *Let $\mathbb{A}$ be an ω-categorical* CSP *template whose relations all have arity at most k and which has local idempotent near unanimity polymorphisms of arity $k + 2$. Then every instance of the* CSP *over $\mathbb{A}$, after enforcing the $(k, k + 1)$-consistency, is sensitive.*

Bounded strict width k of an ω-categorical template was characterized in [10] by the existence of a *quasi-near unanimity* polymorphism n of arity $k + 1$, i.e.,

$$n(y, x, \dots, x) \approx n(x, y, \dots, x) \approx \cdots \approx n(x, x, \dots, y) \approx n(x, x, \dots, x),$$

which is, additionally, *oligopotent*, i.e., the unary operation $x \mapsto n(x, x, \dots, x)$ is equal to an automorphism on every finite set. This result extends the characterization of Feder and Vardi since an oligopotent quasi-near unanimity polymorphism generates a near unanimity polymorphism as soon as the domain is finite. On an infinite domain, however, oligopotent quasi-near unanimity polymorphisms generate local near unanimity polymorphisms which, unfortunately, do not need to be idempotent on the whole domain. Our results thus fall short of proving the following natural generalization of Corollary 16 to the infinite.

▶ **Conjecture 18.** *Let $\mathbb{A}$ be an ω-categorical* CSP *template whose relations all have arity at most k and which has an oligopotent quasi-near unanimity polymorphism of arity $k+2$. Then every instance of the* CSP *over $\mathbb{A}$, after enforcing the $(k, k + 1)$-consistency, is sensitive.*

To confirm the conjecture, a new approach, that does not use a loop lemma, will be needed since there are examples of ω-categorical structures having oligopotent quasi-near unanimity polymorphisms for which the counterpart to Theorem 26 does not hold. Indeed, one such an example is the infinite clique.

References

1. Kirby A. Baker and Alden F. Pixley. Polynomial interpolation and the Chinese remainder theorem for algebraic systems. *Math. Z.*, 143(2):165–174, 1975. `doi:10.1007/BF01187059`.
2. Libor Barto. Finitely related algebras in congruence distributive varieties have near unanimity terms. *Canad. J. Math.*, 65(1):3–21, 2013. `doi:10.4153/CJM-2011-087-3`.

3 Libor Barto. The collapse of the bounded width hierarchy. *J. Logic Comput.*, 26(3):923–943, 2016. doi:10.1093/logcom/exu070.
4 Libor Barto and Marcin Kozik. Congruence distributivity implies bounded width. *SIAM J. Comput.*, 39(4):1531–1542, December 2009. doi:10.1137/080743238.
5 Libor Barto and Marcin Kozik. Constraint satisfaction problems solvable by local consistency methods. *J. ACM*, 61(1):3:1–3:19, January 2014. doi:10.1145/2556646.
6 Libor Barto and Marcin Kozik. Absorption in Universal Algebra and CSP. In Andrei Krokhin and Stanislav Zivny, editors, *The Constraint Satisfaction Problem: Complexity and Approximability*, volume 7 of *Dagstuhl Follow-Ups*, pages 45–77. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, Dagstuhl, Germany, 2017. doi:10.4230/DFU.Vol7.15301.45.
7 Libor Barto, Andrei Krokhin, and Ross Willard. Polymorphisms, and How to Use Them. In Andrei Krokhin and Stanislav Zivny, editors, *The Constraint Satisfaction Problem: Complexity and Approximability*, volume 7 of *Dagstuhl Follow-Ups*, pages 1–44. Schloss Dagstuhl–Leibniz-Zentrum für Informatik, Dagstuhl, Germany, 2017. doi:10.4230/DFU.Vol7.15301.1.
8 George M. Bergman. On the existence of subalgebras of direct products with prescribed d-fold projections. *Algebra Universalis*, 7(3):341–356, 1977. doi:10.1007/BF02485443.
9 Manuel Bodirsky. Complexity classification in infinite-domain constraint satisfaction. Mémoire d'habilitation à diriger des recherches, Université Diderot – Paris 7, 2012. URL: http://arxiv.org/abs/1201.0856.
10 Manuel Bodirsky and Víctor Dalmau. Datalog and constraint satisfaction with infinite templates. *Journal of Computer and System Sciences*, 79(1):79 – 100, 2013. doi:https://doi.org/10.1016/j.jcss.2012.05.012.
11 H. Chen, M. Valeriote, and Y. Yoshida. Testing assignments to constraint satisfaction problems. In *2016 IEEE 57th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 525–534, Oct 2016. doi:10.1109/FOCS.2016.63.
12 Tomás Feder and Moshe Y. Vardi. The computational structure of monotone monadic SNP and constraint satisfaction: a study through Datalog and group theory. *SIAM J. Comput.*, 28(1):57–104 (electronic), 1999. doi:10.1137/S0097539794266766.
13 Jonah Horowitz. Computational complexity of various Mal'cev conditions. *Internat. J. Algebra Comput.*, 23(6):1521–1531, 2013. doi:10.1142/S0218196713500343.
14 András Huhn. Schwach distributive Verbände. *Acta Fac. Rerum Natur. Univ. Comenian. Math.*, pages 51–56, 1971.
15 Peter Jeavons, David Cohen, and Martin C. Cooper. Constraints, consistency and closure. *Artificial Intelligence*, 101(1-2):251–265, 1998. doi:10.1016/S0004-3702(98)00022-8.
16 Emil W. Kiss and Péter Pröhle. Problems and results in tame congruence theory. A survey of the '88 Budapest Workshop. *Algebra Universalis*, 29(2):151–171, 1992. doi:10.1007/BF01190604.
17 Marcin Kozik, Andrei Krokhin, Matt Valeriote, and Ross Willard. Characterizations of several Maltsev conditions. *Algebra Universalis*, 73(3-4):205–224, 2015. doi:10.1007/s00012-015-0327-2.
18 Miroslav Olšák. The weakest nontrivial idempotent equations. *Bulletin of the London Mathematical Society*, 49(6):1028–1047, 2017. doi:10.1112/blms.12097.
19 Michael Pinsker. Algebraic and model theoretic methods in constraint satisfaction. arXiv:1507.00931, 2015.
20 Ágnes Szendrei. Rosenberg-type completeness criteria for subclones of Slupecki's clone. *Proceedings of The International Symposium on Multiple-Valued Logic*, pages 349–354, 2012. doi:10.1109/ISMVL.2012.54.

A Proofs of Theorems 6 and 9

The first result is due to Bergman [8], we provide a short proof for the convenience of the reader.

▶ **Theorem 6.** *Let $k > 1$ and $\mathcal{V}$ be a variety. The following are equivalent:*
1. *$\mathcal{V}$ has a $(k+1)$-ary near unanimity term;*
2. *any partial solution of a $(k, k+1)$-instance over $\mathcal{V}$ extends to a solution.*

Proof of Theorem 6. A straightforward modification of the "if direction" of the proof of Theorem 7, using Proposition 13 in place of Proposition 14 shows that the second condition implies the existence of a $(k+1)$-ary near unanimity term (also see [8, Lemma 11]). For the converse, suppose that $\mathcal{V}$ has a $(k+1)$-ary near unanimity term $n(x_1, \ldots, x_{k+1})$ and let $\mathcal{I} = (V, \mathcal{C})$ be a $(k, k+1)$-instance of $\mathsf{CSP}(\mathcal{V})$.

Let $n = |V|$. We will show by induction on $r < n$ that if $W \subseteq V$ with $|W| = r$ then any solution of $\mathcal{I}_{|W}$ can be extended to a solution of $\mathcal{I}_{|W \cup \{v\}}$ for any $v \in V \setminus W$. From this, the implication will follow. By the assumption that $\mathcal{I}$ is a $(k, k+1)$-instance it follows that this property holds for $r = k$. So, assume that $k < r < n$ and suppose that $W \subseteq V$ with $|W| = r$. Let $v \in V \setminus W$ and let f be a solution of $\mathcal{I}_{|W}$.

Fix some listing of the elements of W, say $W = \{v_1, v_2, \ldots, v_r\}$ and for $1 \leq i \leq r$ let $W_i = (W \setminus \{v_i\}) \cup \{v\}$. By induction, there is a solution f_i of $\mathcal{I}_{|W_i}$ that extends the restriction of f to $W \setminus \{v_i\}$, for $1 \leq i \leq k+1$. We claim that the extension of f to $W \cup \{v\}$ by setting $f(v) = n(f_1(v), f_2(v), \ldots, f_{k+1}(v))$ produces a solution of $\mathcal{I}_{|W \cup \{v\}}$.

We need to show that if $U \subseteq W \cup \{v\}$ with $|U| = k$ then $(f(u) \mid u \in U)$ satisfies the unique constraint (U, R) of $\mathcal{I}$ with scope U. When $U \subseteq W$, this is immediate, so assume that $v \in U$. For $1 \leq i \leq k+1$, let g_i be the restriction of f_i to U, if $v_i \notin U$ and otherwise let g_i be some partial solution of $\mathcal{I}_{|U}$ that extends the restriction of f_i to $U \setminus \{v_i\}$. Since each g_i satisfies the constraint (U, R) then so does $n(g_1, g_2, \ldots, g_{k+1})$. Using that n is a near unanimity term it can be shown that this element is equal to $f_{|U}$, as required. ◀

The next theorem is a variation of the Baker-Pixley [1] result for idempotent, not necessarily finite, algebras.

▶ **Theorem 9.** *Let $\mathbf{A}$ be an idempotent algebra and $k > 1$. The following are equivalent:*
1. *$\mathbf{A}$ has local near unanimity term operations of arity $k+1$;*
2. *for every $r > k$, every subalgebra of $\mathbf{A}^r$ is uniquely determined by its projections onto all k-element subsets of coordinates;*
3. *every $(k+1)$-generated subalgebra of $\mathbf{A}^{k+1}$ is uniquely determined by its projections onto all k-element subsets of coordinates.*

Proof of Theorem 9. To show that Condition 1 implies Condition 2, suppose that $\mathbf{A}$ has local near unanimity term operations of arity $k+1$ and let $\mathbf{R}$ be a subalgebra of $\mathbf{A}^r$ for some $r > k$. Let $\mathbf{a} = (a_1, \ldots, a_r) \in A^r$ be a tuple such that for every subset I of $[r]$ of size k, there is some element $\mathbf{b} \in R$ with $\mathrm{proj}_I(\mathbf{a}) = \mathrm{proj}_I(\mathbf{b})$. We will show by induction on $n \geq k$ that if $n \leq r$ then for every subset J of $[r]$ of size n there is some $\mathbf{b} \in R$ with $\mathrm{proj}_J(\mathbf{a}) = \mathrm{proj}_J(\mathbf{b})$. With $n = r$ we conclude that $\mathbf{a} \in R$, as required.

By assumption, this property holds when $n = k$. Suppose that it has been established for some n with $k \leq n < r$ and let J be a subset of $[r]$ of size $n+1$. By symmetry it suffices to consider the case when $J = \{1, 2, \ldots, n+1\}$. For each i, with $1 \leq i \leq k+1$, let $\mathbf{b}_i \in R$ be such that $\mathbf{a}$ and $\mathbf{b}_i$ agree on the set $J \setminus \{i\}$. Let $n(x_1, \ldots, x_{k+1})$ be a $(k+1)$-ary local near unanimity term operation of $\mathbf{A}$ for the subset of A consisting of all of the components of the tuples $\mathbf{b}_i$, for $1 \le i \le k+1$. A straightforward calculation shows that $\mathbf{b} = n(\mathbf{b}_1, \dots, \mathbf{b}_{k+1}) \in R$ has the desired property.

Clearly Condtion 2 implies Condition 3. For the remaining implication, we use Corollary 2.7 from [13] that shows that if $\mathbf{A}$ is finite (and idempotent) then it will have a $(k+1)$-ary near unanimity term operation if and only if for every $a_i, b_i \in A$, for $1 \le i \le k+1$, there is some term operation t of $\mathbf{A}$ such that

$$\begin{aligned} t(b_1, a_1, a_1, \dots, a_1) &= a_1 \\ t(a_2, b_2, a_2, \dots, a_2) &= a_2 \\ &\vdots \\ t(a_{k+1}, a_{k+1}, a_{k+1}, \dots, b_{k+1}) &= a_{k+1}. \end{aligned}$$

It can be seen from the proof of this result that if $\mathbf{A}$ is not assumed to be finite, then one can conclude that it has local near unanimity term operations of arity $k+1$ if and only if this condition holds for all a_i and b_i.

This local term condition can be translated into a statement about subalgebras of $\mathbf{A}^{k+1}$, namely that for every $a_i, b_i \in A$, for $1 \le i \le k+1$, the $(k+1)$-tuple $\mathbf{a} = (a_1, \dots, a_{k+1})$ belongs to the subalgebra $\mathbf{R}$ of $\mathbf{A}^{k+1}$ generated by the set of $k+1$ tuples

$$\{(b_1, a_2, a_3, \dots, a_{k+1}), (a_1, b_2, a_3, \dots, a_{k+1}), \dots, (a_1, a_2, a_3, \dots, b_{k+1})\}.$$

Our assumption on $\mathbf{A}$ guarantees that $\mathbf{a}$ belongs to R since any projection of R onto k coordinates will contain the corresponding projection of $\mathbf{a}$. Thus $\mathbf{A}$ will have local near unanimity term operations of arity $k+1$. ◀

B Proof of Theorem 26

In this section we present a proof of Theorem 26. The proof is a trivial adaptation of reasoning attributed to Ralph McKenzie in [18].

▶ **Theorem 26.** *Let $\mathbf{A}$ be an idempotent algebra and $\mathbf{R} \le \mathbf{A}^2$ be nonempty and symmetric. If R locally absorbs $=_A$, then R contains a loop.*

The remaining part of this section is devoted to a proof of Theorem 26 by the way of contradiction.

Let n denote the arity of the absorbing operations. We choose a counterexample to the theorem minimal with respect to n. Then, we fix an algebra $\mathbf{A}$ and will call an $\mathbf{R} \le \mathbf{A}^2$ a *counterexample candidate* if it is non-empty, symmetric, locally n-absorbs $=_A$ and has no loop.

▷ Claim 19. Every counterexample candidate has a closed walk of odd length.

Proof. Since R is nonempty and symmetric we have $(a,b), (b,a) \in R$. Apply Lemma 25 to the walk $(a, b, a, b, \dots, a/b)$ of length $n-1$ (i.e., n vertices, $n-1$ steps) taken twice (where the last element is either a or b depending on the parity of n). The lemma provides a directed walk of length n connecting the first and last elements. Since R is symmetric all the edges are undirected and we obtained a closed walk of odd length. ◀

▷ Claim 20. There exists a counterexample candidate containing a 3-element clique.

$$\begin{array}{cccccc} c_1 & c_1 & c_2 & c_3 & \cdots & c_{n-1} \\ | & | & \| & | & & | \\ c_1' & c_1' & c_2 & c_3' & \cdots & c_{n-1}' \\ | & | & \| & | & & | \\ c_1'' & c_1'' & c_2 & c_3'' & \cdots & c_{n-1}'' \\ | & | & \| & | & & | \\ d_1 & d_1 & c_2 & d_3 & \cdots & d_{n-1} \end{array} \longrightarrow \begin{array}{c} t(c_1, c_1, c_2, c_3, \dots, c_{n-1}) \\ | \\ t(c_1', c_1', c_2, c_3', \dots, c_{n-1}') \\ | \\ t(c_1'', c_1'', c_2, c_3'', \dots, c_{n-1}) \\ | \\ t(d_1, d_1, c_2, d_3, \dots, d_{n-1}) \end{array}$$

Figure 1 Solid lines are are S-related and dashed lines are T-related.

Proof. Take a counterexample candidate R; it has an odd cycle, and if it has a triangle we are done. Thus the length of a shortest odd cycle is greater than 3. In this case, however $R \circ R \circ R$ is a counterexample candidate (we use Lemma 24 to provide local absorption) with shorter odd cycle. We proceed this way and, in the end, find a counterexample candidate with a 3-cycle (which is a 3-clique). ◀

▷ Claim 21. No counterexample candidate contains an n-element clique.

Proof. Suppose $(a_1, \dots, a_n)$ is such a clique. We can choose, using the definition of local absorption, t such that $(t(a_1, \dots, a_n), t(a_i, \dots, a_i)) \in R$ for all i. We use this fact, and the fact that $\mathbf{R} \leq \mathbf{A}^2$, to conclude that

$$\big(t(t(a_1, \dots, a_n), \dots, t(a_1, \dots, a_n)), t(a_1, \dots, a_n)\big) \in R,$$

but, by the idempotency of t, the two elements are equal and we have obtained a loop — a contradiction. ◀

In order to finish the proof we fix R to be a counterexample candidate with a 3-element clique and let $a_1, \dots, a_m$ be distinct, forming a maximal clique in R (such a clique exists by the last claim). Let B be the subset of A containing vertices with edges to each of $a_1, \dots, a_{m-2}$. Note that B is a subuniverse (since $\mathbf{A}$ is idempotent) and $S = B^2 \cap R$ is nonempty as $(a_{m-1}, a_m), (a_m, a_{m-1}) \in S$.

Note, that $\mathbf{S} \leq \mathbf{B}^2$ is symmetric, nonempty, has no 3-clique and it locally n-absorbs $=_B$. We obtain a contradiction by showing that $\mathbf{T} = \mathbf{S} \circ \mathbf{S} \circ \mathbf{S}$ locally $n-1$ absorbs $=_B$. The graph T is non-empty, symmetric, has no loop and $\mathbf{T} \leq \mathbf{B}^2$. We will fix a one-$=_B$-in-$T$ tuple, and construct a finite set of one-$=_A$-in-R tuples such that if $t(x_1, \dots, x_n)$ is an operation of $\mathbf{A}$ producing elements of R on the tuples from the last set then $t(x_1, x_1, x_2, x_3, \dots, x_{n-1})$ produces an element of T on the original tuple. The theorem we are working to prove clearly follows from this fact.

Let $(c_1, d_1), (c_2, d_2), \dots, (c_{n-1}, d_{n-1})$ be a one-$=_B$-in-T tuple. We consider two cases: in case one $c_i = d_i$ for some $i > 1$ and in case two $c_1 = d_1$. In case one (see Figure 1), we assume, wlog that $i = 2$, and find for all $j \neq 2$ elements c_j', c_j'' such that $(c_j, c_j'), (c_j', c_j''), (c_j'', d_j) \in S$. It suffices to take care of the three following one-$=_A$-in-R evaluations:

$$(c_1, c_1'), (c_1, c_1'), (c_2, c_2), (c_3, c_3'), \dots, (c_{n-1}, c_{n-1}'),$$
$$(c_1', c_1''), (c_1', c_1''), (c_2, c_2), (c_3', c_3''), \dots, (c_{n-1}', c_{n-1}'') \text{ and}$$
$$(c_1'', d_1), (c_1'', d_1), (c_2, c_2), (c_3'', d_3), \dots, (c_{n-1}'', d_{n-1}).$$

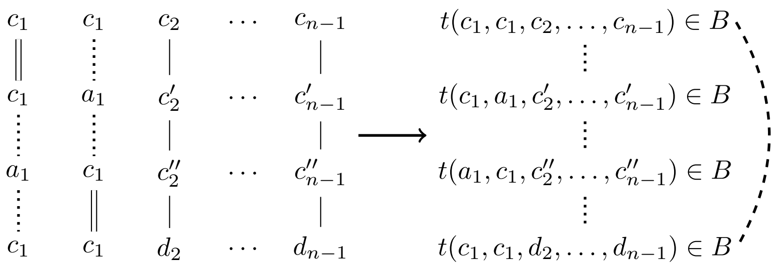

Figure 2 Solid lines are are S-related, dashed lines are T-related, and dotted lines are R-related.

In case two (see Figure 2) the situation is a bit more involved, we define c_i', c_i'' for all $i > 1$ but need 4 evaluations:

$$
\begin{aligned}
&(c_1, c_1), (c_1, a_1), (c_2, c_2') \dots, (c_{n-1}, c_{n-1}'),\\
&(c_1, a_1), (a_1, c_1), (c_2', c_2''), \dots, (c_{n-1}', c_{n-1}''),\\
&(a_1, c_1), (c_1, c_1), (c_2'', d_2), \dots, (c_{n-1}'', d_{n-1}) \text{ and two new ones}\\
&(c_1, a_1), (a_1, a_1), (c_2', a_1), \dots, (c_{n-1}', a_1),\\
&(a_1, a_1), (c_1, a_1), (c_2'', a_1), \dots, (c_{n-1}'', a_1).
\end{aligned}
$$

The list contains 5 evaluations, but the second one (included for simplicity) is in fact not a one-$=_A$-in-R evaluation, but a usual application of the term to elements of R. Any term, putting all these evaluations in R puts (by idempotency and the fact that all considered elements are adjacent to a_i if $1 < i < m-1$) $t(c_1, a_1, c_2', \dots, c_{n-1}'), t(a_1, c_1, c_2'', \dots, c_{n-1}'') \in B$. These elements witness the path required to put the pair $(t(c_1, c_1, c_2, c_3, \dots, c_{n-1}), t(c_1, c_1, d_2, d_3 \dots, d_{n-1}))$ in T.

C New loop lemmata

A *loop lemma* is a theorem stating that a binary relation satisfying certain structural and algebraic requirements necessarily contains a *loop* – a pair (a, a). In this section we provide two new loop lemmata, Theorem 27 and Theorem 28, which generalize an “infinite loop lemma” of Olšák [18] and may be of independent interest. Theorem 28 is a crucial tool for the proof presented in Appendix D.

The algebraic assumptions in the new loop lemmata concern absorption, a concept that proved useful in the algebraic theory of CSPs and Universal Algebra [6]. We adjust the standard definition to our specific purposes. We begin with a very elementary definition.

▶ **Definition 22.** *Let R and S be sets. We call a tuple $(a_1, \dots, a_n)$ a* one-S-in-R tuple *if for exactly one i we have $a_i \in S$ and all the other a_i's are in R.*

Next we proceed to define a relaxation of the standard absorbing notion. We follow a standard notation, silently extending operations of an algebra to powers (by computing them coordinate-wise).

▶ **Definition 23.** *Let $\mathbf{A}$ be an algebra, $\mathbf{R} \leq \mathbf{A}^k$ and $S \subseteq A^k$. We say that R locally n-absorbs S if, for every finite set $\mathcal{C}$ of one-S-in-R tuples of length n, there is an operation t of $\mathbf{A}$ such that $t(\mathbf{a^1}, \dots, \mathbf{a^n}) \in R$ whenever $(\mathbf{a^1}, \dots, \mathbf{a^n}) \in \mathcal{C}$. We will say that R* locally absorbs *S, if R locally n-absorbs S for some n.*

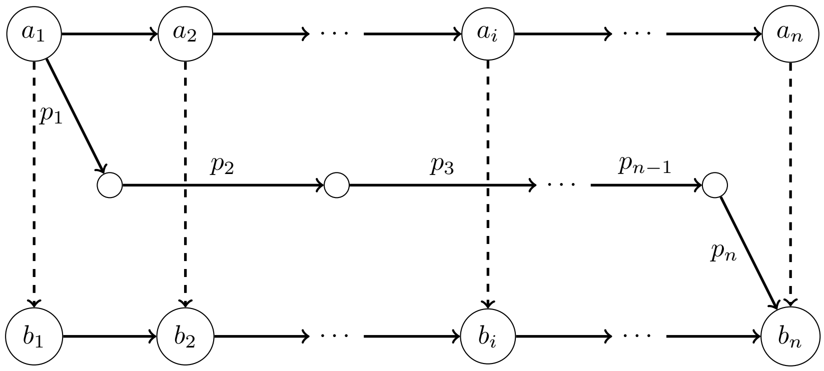

Figure 3 Solid arrows represent tuples from R and dashed arrows represent tuples from S.

Absorption, even in our monstrous form, is stable under various constructions. The following lemma lists some of them and we leave it without a proof (the reasoning is identical to the one in e.g. Proposition 2 in [6]).

▶ **Lemma 24.** *Let $\mathbf{A}$ be an algebra, $\mathbf{R} \leq \mathbf{A}^2$ and R locally n-absorbs S. Then R^{-1} locally n-absorbs S^{-1}; and $R \circ R$ locally n-absorbs $S \circ S$, and $R \circ R \circ R$ locally n-absorbs $S \circ S \circ S$ etc.*

Let us prove a first basic property of local absorption.

▶ **Lemma 25.** *Let $\mathbf{A}$ be an idempotent algebra, $\mathbf{R} \leq \mathbf{A}^2$ and R locally n-absorbs S. Let $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ be directed walks in R, and let $(a_i, b_i) \in S$ for each i (see Figure 3). Then there exists a directed walk from a_1 to b_n of length n in R.*

Proof. We will show that there is an operation t of the algebra $\mathbf{A}$ such that the following $(n+1)$-tuple of elements of A is a walk of length n in R from a_1 to b_n.

$$\begin{aligned}
(a_1 =& t(a_1, a_1, a_1, \ldots, a_1), \\
& t(b_1, a_2, a_2, \ldots, a_2), \\
& t(b_2, b_2, a_3, \ldots, a_3), \\
& \vdots \\
& t(b_{n-1}, b_{n-1}, \ldots, b_{n-1}, a_n), \\
b_n =& t(b_n, b_n, b_n, \ldots, b_n)).
\end{aligned}$$

In order to choose a proper t we apply the definition of local absorption to the set of $(n+1)$ one-S-in-R tuples corresponding to the steps in the path. ◀

The loop lemma of Olšák concerns symmetric relations absorbing the equality relation $\{(a, a) \mid a \in A\}$, which is denoted $=_A$. The original result, stated in a slightly different language, does not cover the case of local absorption. However, a typographical modification of a proof mentioned in [18] shows that the theorem holds. For completeness sake, we present this proof in Appendix B.

▶ **Theorem 26** ([18]). *Let $\mathbf{A}$ be an **idempotent** algebra and $\mathbf{R} \leq \mathbf{A}^2$ be nonempty and symmetric. If R locally absorbs $=_A$, then R contains a loop.*

In order to apply this theorem in the case of sensitive instances, we need to generalize it. In the following two theorems we will gradually relax the requirement that $\mathbf{R}$ is symmetric. In the first step, we substitute it with a condition requiring a closed, directed walk in the graph (i.e., a sequence of possibly repeating vertices, with consecutive vertices connected by forward edges and the first and last vertex identical). Recall that R^{-1} is the inverse relation to R and let us denote by $R^{\circ l}$ the l-fold relational composition of R with itself.

▶ **Theorem 27.** *Let* $\mathbf{A}$ *be an* ***idempotent*** *algebra and* $\mathbf{R} \leq \mathbf{A}^2$ *contain a directed closed walk. If* $\mathbf{R}$ *locally absorbs* $=_A$*, then* R *contains a loop.*

Proof. Let n denote the arity of the absorbing operations. The proof is by induction on $l \geq 0$, where l is a number such that there exists a directed closed walk from a_1 to a_1 of length 2^l.

We start by verifying that such an l exists. Take a directed walk $(a_1, \dots, a_{k-1}, a_k = a_1)$ in R. We may assume that its length k is at least n, since we can, if necessary, traverse the walk multiple times. An application of Lemma 25 to the relations $R, =_A$ and tuples $(a_1, \dots, a_n)$, $(a_1, \dots, a_n)$ gives us a directed walk from a_1 to a_n of length n. Appending this walk with the walk $(a_n, a_{n+1}, \dots, a_k = a_1)$ yields a directed walk from a_1 to a_1 of length $k+1$. In this way, we can get a directed walk from a_1 to a_1 of any length greater than k.

Now we return to the inductive proof and start with the base of induction for $l = 0$ or $l = 1$. If $l = 0$, then we have found a loop. If $l = 1$ we have a closed walk of length 2, that is, a pair (a, b) which belongs to both R and R^{-1}. We set $R' = R \cap R^{-1}$ and observe that R' is nonempty and symmetric, and it is not hard to verify that R' locally absorbs $=_A$. Olšák's loop lemma, in the form of Theorem 26, gives us a loop in R.

Finally, we make the induction step from $l - 1$ to l. Take a closed walk $(a_1, a_2, \dots)$ of length 2^l and consider $R' = R^{\circ 2}$. Observe that R' contains a directed closed walk of length 2^{l-1} (namely $(a_1, a_3, \dots)$), and that R' locally absorbs $=_A$ (by Lemma 24), so, by the inductive hypothesis, R' has a loop. In other words, R has a directed closed walk of length 2 and we are done by the case $l = 1$. ◀

Note that we cannot further relax the assumption on the graph by requiring that, for example, it has an infinite directed walk. Indeed the natural order of the rationals (taken for R) locally 2-absorbs the equality relation by the binary arithmetic mean operation $(a + b)/2$ (i.e. all the absorbing evaluations are realized by a single operation). The same relation locally 4-absorbs equality with the near unanimity operation $n(x, y, z, w)$ which, when applied to $a \leq b \leq c \leq d$, in any order, returns $(b + c)/2$.

Nevertheless, we can strengthen the algebraic assumption and still provide a loop; the following theorem is one of the key components in the proof of Theorem 43 (albeit applied there with $l = 1$).

▶ **Theorem 28.** *Let* $\mathbf{A}$ *be an* ***idempotent*** *algebra and* $\mathbf{R} \leq \mathbf{A}^2$ *contain a directed walk of length* $n - 1$*. If* R *locally* n*-absorbs* $=_A$ *and* $R^{\circ l}$ *locally* n*-absorbs* R^{-1} *for some* $l \in \mathbb{N}$ *then* R *contains a loop.*

Proof. By applying Lemma 25 similarly as in the proof of Theorem 27, we can get, from a directed walk of length $n - 1$, a directed walk $(a_1, a_2, \dots)$ of an arbitrary length. Moreover, by the same reasoning, for each i and j with $j \geq i + n - 1$, there is a directed walk from a_i to a_j of any length greater than or equal to $j - i$.

Consider the relations $R' = R^{\circ l n^2}$ and $S = (R^{-1})^{\circ n^2}$, and tuples

$$\mathbf{c} = (c_1, \dots, c_n) := (a_{n^2}, a_{(n+1)n}, \dots a_{(2n-1)n}), \text{ and}$$
$$\mathbf{d} = (d_1, \dots, d_n) := (a_n, a_{2n} \dots, a_{n^2})$$

By the previous paragraph and the definitions, both $\mathbf{c}$ and $\mathbf{d}$ are directed walks in R', and $(c_i, d_i) \in S$ for each i. Moreover, since $R^{\circ l}$ locally n-absorbs R^{-1}, Lemma 24 implies that R' locally absorbs S. We can thus apply Lemma 25 to the relations R', S and the tuples $\mathbf{c}, \mathbf{d}$ and obtain a directed walk from $c_1 = a_{n^2}$ to $d_{n-1} = a_{n^2}$ in R'. This closed walk in turn gives a closed directed walk in R and we are in a position to finish the proof by applying Theorem 27. ◀

D Consistent instances are sensitive

In this section we provide a proof for the "only if direction" in Theorem 7 and "1 implies 2" in Theorem 11. We will proceed with the two proofs in parallel; in one case we fix an algebra $\mathbf{A}$ and in the other a variety $\mathcal{V}$. We will assume, without loss of generality, that the only operation symbol of $\mathcal{V}$ is $(k+2)$-ary and is a near unanimity operation for all members of $\mathcal{V}$. So, all members of $\mathcal{V}$ are idempotent. Formally, in the case of Theorem 11, we should be working with instances over $\mathbf{A}^2$, but if $\mathbf{A}$ has local $(k+2)$-ary near unanimity term operations, then so does $\mathbf{A}^2$ and so we can work directly with an algebra possessing local near unanimity term operations and denote it by $\mathbf{A}$. We will remark on the differences between these two cases only in the places where we apply near unanimity operations.

For the purpose of this section we modify the definition of an instance slightly: an instance is a triple $\mathcal{I} = (V, \{\mathbf{A}_x \mid x \in V\}, \mathcal{C})$, where $\mathcal{C} = \{(S, R_S) \mid S \subseteq V, |S| \leq k\}$ and $R_S \leq \prod_{x \in S} \mathbf{A}_x$. Note that the definition of a CSP instance is, formally, different than our standard definition: the variables involved in a constraint are a set and not a tuple. This minor modification will allow us to present the proofs more succinctly. In order for the interpretation of a constraint to be unique we assume, without loss of generality, that the algebras $\mathbf{A}_x$ are disjoint. When applying the results of this section in Theorem 7 we will set each $\mathbf{A}_x$ to be an isomorphic copy of $\mathbf{A}$, and in case of Theorem 11 we will choose isomorphic copies from the variety, so that their domains are disjoint.

The rough idea of the proof is to fix, in a $(k, k+1)$-instance, a tuple from the relation constraining set of variables Y and consider the instance obtained by removing Y from the set of variables and shrinking the constraint relations so that only the tuples extending the fixed choice of values for the variables in Y remain. If we were able to show that the obtained instance contains a $(k, k+1)$-subinstance, both theorems would then follow by induction on the number of variables of the instance. It is well known that for instances **with finite domains**, the latter property is equivalent to the solvability of certain relaxed instances, here called k-trees. Our strategy for the proof is, in fact, to prove the solvability of k-trees, by induction on a measure of complexity of k-trees. Unfortunately, for infinite domains, the solvability of k-trees is in general weaker than having a $(k, k+1)$-subinstance, and this brings several technical complications into our proof. In particular, we will be working with CSP instances, that **won't necessarily be** $(k, k+1)$**-instances**, or even k-uniform.

The remaining parts of this section are organized as follows. In the first subsection we introduce concepts that are useful for working with instances and their solutions – patterns and realizations. The next subsection studies solvability with a fixed evaluation for k-variables and provides two core technical claims for the inductive proof of the solvability of k-trees; the proof is then assembled in the third subsection and the missing parts of Theorems 7 and 11 are derived as a consequence.

Until Theorem 43 in the last section we fix

- an integer $k \geq 2$,
- a variety $\mathcal{V}$ with a $(k+2)$-ary near unanimity term in case of Theorem 7 or an algebra $\mathbf{A}$

with local near unanimity term operations of arity $k+2$ in case of Theorem 11;

- an instance $\mathcal{I} = (V, \{\mathbf{A}_x \mid x \in V\}, \mathcal{C})$, where $\mathcal{C} = \{(S, R_S) \mid S \subseteq V, |S| \leq k\}$ and $R_S \leq \prod_{x\in S} \mathbf{A}_x$, such that, for any $S' \subseteq S$ with $|S| \leq k$, the projection of R_S onto S' is contained in $R_{S'}$ (here either every $\mathbf{A}_x$ is in a variety $\mathcal{V}$ in case of Theorem 7, or $\mathbf{A}_x$ is an isomorphic copy of $\mathbf{A}$ in case of Theorem 11).

A $(k, k+1)$-instance can naturally be expanded to meet the condition in the last item by adding the constraints $(S', R_{S'})$ for $|S'| < k$, where $R_{S'}$ is defined as the projection of R_S onto S' for an arbitrary k-element superset S of S'. It is an easy exercise, and we leave it to the reader, to verify that this definition does not depend on the choice of S.

For a tuple of (not necessarily distinct) variables $x_1, \ldots, x_l$ with $l \leq k$ we denote $R_{x_1,\ldots,x_l} = \{(r_{x_1}, \ldots, r_{x_l}) \mid \mathbf{r} \in R_{\{x_1,\ldots,x_l\}}\} \leq \prod_{i=1}^{l} \mathbf{A}_{x_i}$. Finally, we set $A = \bigcup_{x\in V} A_x$.

D.1 Patterns

A pattern is a hypergraph whose vertices are labeled by variables and hyperedges indicate that constraints should be satisfied. It will be convenient to have the set of hyperedges closed under taking subsets.

▶ **Definition 29.** *A* pattern *is a triple $\mathbb{P} = (P; \mathcal{F}, v)$, where P is a set of* vertices*, $\mathcal{F}$ is a family of at most k-element subsets of P closed under taking subsets, and v is a mapping $v : P \to V$. Members of $\mathcal{F}$ are called* faces *and the variable $v(i)$ is referred to as the* label *of i.*

A realization *of $\mathbb{P}$ is a mapping $\alpha : P \to A$, which is* consistent with v*, that is, $\alpha(i) \in A_{v(i)}$ for every $i \in P$, and* satisfies *every face $\{f_1, \ldots, f_l\} \in \mathcal{F}$, that is, $(\alpha(f_1), \ldots, \alpha(f_l)) \in R_{v(f_1),\ldots,v(f_l)}$.*

For clarity, we will always call a mapping from a set of vertices to A (which is not necessarily a realization of a pattern) an *assignment* (denoted α, β, …), a mapping from a set of variables to A an *evaluation* (denoted ϕ, ρ, …), and a mapping from a set of vertices to V a *labeling* (denoted v). We say that an assignment α *extends* an evaluation ϕ if $\alpha(p) = \phi(v(p))$ for any p in the domain of α such that $v(p)$ is in the domain of ϕ.

Since we assume that the A_x's are disjoint, any assignment uniquely determines a consistent labeling and it makes sense to say that an assignment satisfies a set of vertices F, provided $|F| \leq k$. Also note that, by the assumptions on $\mathcal{I}$, if an assignment α satisfies F, then it satisfies every subset of F. Finally, note that in the same situation $\alpha(i) = \alpha(i')$ whenever $v(i) = v(i')$.

A pattern $\mathbb{P}' = (P'; \mathcal{F}', v')$ is a *subpattern* of $\mathbb{P}$ if $P' \subseteq P$, $\mathcal{F}' \subseteq \mathcal{F}$, and v' is the restriction of v to P'. By a union of two patterns we mean the set-theoretical union of the vertex sets, face sets, and labelings. It can only be formed if there are no collisions among labels.

The richest patterns are the *complete patterns*, whose faces are all the subsets of the vertex set of size at most k. Note that a realization of a complete pattern with $l \leq k$ vertices is essentially the same as a tuple in the corresponding constraint relation. The most important patterns for our purposes are l-trees with $l \leq k$. These are, informally, patterns obtained from the empty pattern by gradually adding complete patterns with at most $l+1$ vertices and merging them along a face to the already constructed pattern.

▶ **Definition 30.** *Let $l \leq k$ and let F be a set of labeled vertices. The* complete l-tree with base F of depth 1 *is the complete pattern with vertex set F. The* complete l-tree with base F of depth $d+1$ *is obtained from the complete l-tree $\mathbb{P}$ with base F of depth d by adding to $\mathbb{P}$,* *for every face E of $\mathbb{P}$ and every $(l+1-|E|)$-element set of variables U, a set G of $|U|$ fresh vertices labeled by all elements of U and all the at most l-element subsets of $E \cup G$ as faces.*

*An l-*tree *is a subpattern of a complete l-tree.*

The significance of l-trees is apparent from the following observation.

▶ **Lemma 31.** *Assume that $\mathcal{I}$ is a $(k, k+1)$-instance (with small arity constraints added). Let $l \leq k$, let $\mathbb{P}$ be an l-tree, and let F be a face of $\mathbb{P}$. Then any assignment $\alpha : F \to A$ that satisfies F can be extended to a realization of $\mathbb{P}$. In particular, every l-tree is realizable.*

Proof. If $\mathbb{P}$ is a complete l-tree with base F, then α can be gradually extended to a realization of $\mathbb{P}$ by a straightforward application of the definition of $(k, k+1)$ instance. It remains to observe that every l-tree with a face F is a subpattern of a complete l-tree with base F. ◀

As noted above, realizability of k-trees in some sense even characterizes $(k, k+1)$ instances for finite domains. From this perspective it makes sense to use k-trees to measure the consistency level (called the quality) of a tuple in a constraint relation and, more generally, the consistency level of a realization.

▶ **Definition 32.** *Let F be a labeled set of vertices of size at most k. We say that an assignment α, whose domain includes F and which is consistent with the labeling,* satisfies F with quality d *if $\alpha_{|F}$ can be extended to a realization of the complete k-tree with base F of depth d. A realization α of a pattern $\mathbb{P}$* has quality d *(or α satisfies $\mathbb{P}$ with quality d) if α satisfies each face of the pattern with quality d.*

Similarly, we say that an evaluation $\phi : W \to A$ (where $|W| \leq k$) has quality d *if the corresponding assignment for a $|W|$-element set of vertices labeled by all the elements of W has quality d.*

Informally, an evaluation ϕ has quality d if it survives d steps in a certain naturally defined consistency procedure. Note that a realization of a pattern is the same as a realization of quality 1 and a realization of quality d is also a realization of quality d' for any $d' \leq d$. Finally, observe that if an assignment α satisfies F with quality d, then it satisfies every subset of F with quality d.

We finish this subsection with two observations.

▶ **Lemma 33.** *The set of quality-d realizations of a pattern $\mathbb{P}$ is a subuniverse of $\prod_{i \in P} \mathbf{A}_{v(i)}$.*

Proof. For d=1 the claim is a straightforward consequence of the fact that constraint relations are subuniverses of products of $\mathbf{A}_x$'s. Otherwise we observe that the set of quality-d realizations of $\mathbb{P}$ is the projection of the set of quality-1 realizations of a larger pattern $\mathbb{Q}$ to P. Indeed, $\mathbb{Q}$ can be taken as the pattern obtained from $\mathbb{P}$ by appending to every face F the complete k-tree with base F of depth d. ◀

▶ **Lemma 34.** *Let $E \subseteq F$ be labeled sets of vertices, $E \leq k$, $|F| \leq k+1$, and let $\alpha : E \to A$ be an assignment which is consistent with the labeling and satisfies E with quality $d+1$. Then α can be extended to an assignment $\beta : F \to A$ which is consistent with the labeling and satisfies each at most k element subset of F with quality d.*

More generally, for any k-tree $\mathbb{P}$, any face F, and any d, there exists d' such that every assignment $\alpha : F \to A$ which satisfies F with quality d' can be extended to a realization of $\mathbb{P}$ of quality d.

Proof. The first observation follows from the definitions while the second one is proved by induction from the first one. ◀

D.2 Fixing patterns

A fixing pattern is a pattern together with a specified set Y of fixing variables. The idea is to require that any consistent evaluation of Y can be extended to a realization of the whole pattern. Since our instance isn't necessarily a $(k, k+1)$-instance the following modification is needed.

▶ **Definition 35.** *A* fixing pattern *is a pair* $(\mathbb{P}, Y)$*, where* $\mathbb{P}$ *is a pattern and* Y *is a set of variables of size at most* k*. The elements of* Y *are called* fixing variables*, the remaining variables from* $v(P) \setminus Y$ *are called* inner*.*

A fixing pattern $(\mathbb{P}, Y)$ *is* f-realizable *if for every* d *there exists* $d' = z_{(\mathbb{P},Y)}(d) \geq d$ *such that every evaluation* $\phi : Y \to A$ *of quality* d' *can be extended to a realization of* $\mathbb{P}$ *of quality* d*.*

It will be a feature of the proofs in this subsection that the sufficient $d' = z_{(\mathbb{P},Y)}(d)$ from the definition will actually depend only on the "shape" of the fixing pattern: it will not depend on the instance, or on the variety, or on the concrete choice of labeling (i.e., the same d' will work for a pattern obtained from $\mathbb{P}$ by changing v to rv for any $r : V \to V$).

A vertex f of a fixing pattern $(\mathbb{P}, Y)$ is called fixing/inner if the variable $v(f)$ is. Faces consisting entirely of inner variables are called *inner*, the remaining faces are called *fixing.* A fixing face, whose set of inner vertices is F and whose set of labels of fixing vertices is Y', is denoted $[F, Y']$. Note that the definition of f-realization only depends on the "inner part" of the fixing pattern together with the list of those $[F, Y']$ that are present in the fixing pattern. It will often be convenient to choose $\mathbb{P}$ *free*, that is, the sets of fixing vertices of any two maximal fixing faces are disjoint.

An inner face F is called *completely fixed* if $[F, Y']$ is a (fixing) face for every $(k - |F|)$-element set of variables $Y' \subseteq Y$. If $\mathbb{Q}$ is a pattern and Y a set of variables of size at most k, which is disjoint from $v(Q)$, then the *complete* Y*-fixing* (*complete vertex* Y*-fixing*, respectively) of $\mathbb{Q}$ is the free fixing pattern $(\mathbb{P}, Y)$, whose set of inner faces coincides with the set of faces of $\mathbb{Q}$ and each inner face (inner vertex, respectively) is completely fixed. Since complete fixings are chosen freely, a complete fixing of a k-tree is a k-tree.

We say that a pattern $\mathbb{Q}$ is *strongly realizable* if each complete fixing of $\mathbb{Q}$ is f-realizable.

Our aim, and the main technical contribution of this section is to prove that every k-tree is strongly realizable. We now present, in Lemma 36 and Lemma 39, two constructions that preserve f-realizability. A proof that the complete fixing of every k-tree can be obtained by these constructions is contained in the next subsection.

▶ **Lemma 36.** *Let* $1 \leq l \leq k+1$*. Let* $(\mathbb{P}, Y)$ *be the complete vertex* Y*-fixing of a complete pattern* $\mathbb{S}$ *with* l *vertices and, if* $l \leq k-1$*, freely add to* $\mathbb{P}$ *an additional fixing face* $[S, Y']$ *(and its subfaces) for some* $Y' \subseteq Y$ *of size* $k - l$*.*

If each complete pattern with $l-1$ *vertices is strongly realizable, then* $(\mathbb{P}, Y)$ *is f-realizable.*

Proof. The case $l = 1$ follows directly from Lemma 34 with the choice $d' = d + 1$ and we henceforth assume $l > 1$.

Fix an arbitrary d. We need to choose d' large enough so that the applications of the assumptions or Lemma 34, which will be used in the proof, do not decrease the quality of our assignments below d. Specifically, we first choose d'' so that $d'' \geq d+2$ and, in case that $l = k+1$, also $d'' \geq z_{(\mathbb{Q},Z)}(d+1)$ for each complete fixing $(\mathbb{Q}, Z)$ of a complete pattern with 2 vertices; and then choose d' so that $d' \geq z_{(\mathbb{Q},Z)}(d'')$ for each complete fixing $(\mathbb{Q}, Z)$ of a

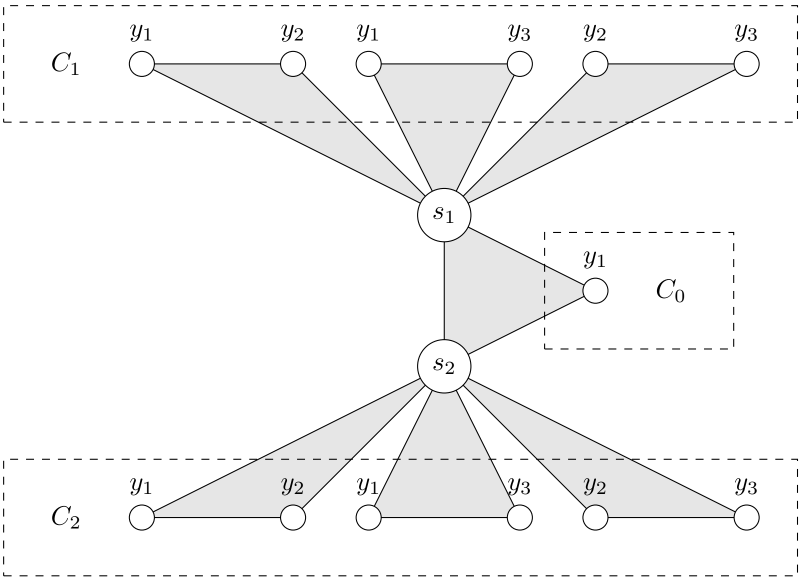

Figure 4 Case $k = 3$, $l = 2$ in the proof of Lemma 36.

complete pattern with $l - 1$ vertices (we will actually only use $(\mathbb{Q}, Z)$ equal to $(\mathbb{P}, Y)$ take away one inner vertex).

Denote $S = \{s_1, \ldots, s_l\}$ the set of inner vertices of $(\mathbb{P}, Y)$, C_i (where $i \in [l]$) the set of fixing vertices coming from the vertex-fixing faces $[\{s_i\}, \ldots]$, and C_0 the set of fixing vertices coming from the fixing face $[S, Y']$ (which is empty if $l \geq k$), see Figure 4. Let $C = C_0 \cup C_1 \cup \cdots \cup C_l$. Finally, let $\phi : Y \to A$ be an evaluation of quality d'.

We consider the set T of restrictions of quality-d realizations of $\mathbb{P}$ to the set C. Note that this set is a subuniverse of the product of the corresponding $\mathbf{A}_x$'s by Lemma 33.

$$T = \{\beta_{|C} : \beta \text{ satisfies } \mathbb{P} \text{ with quality } d\} \leq \prod_{c \in C} \mathbf{A}_{v(c)}$$

We need to prove that the tuple $\mathbf{a}$ defined by $\mathbf{a}(c) = \phi(v(c))$ for all $c \in C$ is in T. By the Baker-Pixley Theorem (Theorem 4 when proving Theorem 7 and Theorem 9 when proving Theorem 11) it is enough to show that for any $(k+1)$-element set of coordinates D, the relation T contains a tuple $\mathbf{b}$ that agrees with $\mathbf{a}$ on this set. This is now our aim.

Denote $D_i = C_i \cap D$ and assume that there exists $i \geq 1$ such that $|D_0 \cup D_i| \leq k - l + 1$. In this case we find a suitable tuple $\mathbf{b}$ in three steps as follows. First, by the choice of d', we can extend ϕ to an assignment $\gamma : P \setminus \{s_i\} \to A$ that satisfies every k-element subset of $P \setminus \{s_i\}$ with quality d'', and set $\beta(p) = \gamma(p)$ for each $p \in P \setminus (\{s_i\} \cup C_0 \cup C_i)$. Second, set $\beta(p) = \phi(v(p))$ for each $p \in D_0 \cup D_i$, let $F = (S \setminus \{s_i\}) \cup D_0 \cup D_i$, and note that F has size at most $(l-1) + (k-l+1) = k$ and that β satisfies F with quality d''. Therefore, by Lemma 34, $\beta_{|F}$ can be extended to $F \cup \{s_i\}$ so that β satisfies each at most k-element subset of $F \cup \{s_i\}$ with quality $d'' - 1 \geq d + 1$. Third, for each face E of $\mathbb{P}$ where β is not yet fully defined we again use Lemma 34 and extend $\beta_{|E \cap \mathrm{dom}(\beta)}$ to E so that β satisfies E with quality d. By construction, $\beta(c) = \phi(v(c))$ for every $c \in D$, and β satisfies every face of $\mathbb{P}$ with quality d: the fixing faces within $P \setminus (C_0 \cup C_i)$ because of the first step, the face S because of the second step, and the remaining fixing faces (within $S \cup C_0 \cup C_i$) because of the third step. Therefore $\mathbf{b} = \beta_{|C}$ is from T and agrees with $\mathbf{a}$ on D.

Let $i \geq 1$ be such that $|D_i|$ is minimal. If $l \leq k$, then simple arithmetic gives us that

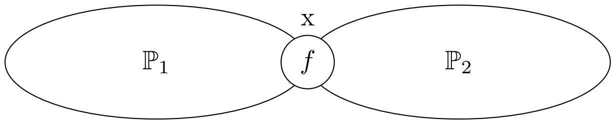

Figure 5 Pattern $\mathbb{Q}$ in Lemma 37.

$|D_0 \cup D_i| \leq k - l + 1$ (so we are done in this case). Indeed, otherwise $|D_i| \geq k - l + 2 - |D_0|$ and $|D| \geq |D_0| + l|D_i| \geq |D_0| + l(k - l + 2 - |D_0|)$. For the maximum size of D_0, that is, $|D_0| = |C_0| = k - l$, the right hand side of the last inequality is equal to $k + l$, and if $|D_0|$ decreases it gets bigger. Then $|D| \geq k + l > k + 1$, a contradiction.

The remaining case is $l = k + 1$ (in particular, $C_0 = D_0 = \emptyset$) and $|D_i| > k - l + 1 = 0$ for each $1 \leq i \leq k+1$. Then, in fact, $D_i = \{d_i\}$ for each $i \geq 1$ (as $|D| \leq k+1$). By the pigeonhole principle, there are $i \neq j$ such that $v(d_i) = v(d_j)$. In this case we modify the three step procedure for finding $\mathbf{b}$ as follows. In the first step we define β only on $P \setminus (\{s_i, s_j\} \cup C_i \cup C_j)$, in the second step we set $\beta(d_i) = \beta(d_j) = \phi(v(d_i))$, define $F = (S \setminus \{s_i, s_j\}) \cup D_i \cup D_j$, and instead of Lemma 34 we use the choice of d'' (coming from complete fixings of 2-element complete patterns) to extend $\beta_{|F}$ to $F \cup \{s_i, s_j\}$. ◀

The next lemma provides the base case for the second construction. We remark that having a near unanimity term of arity $2k$, when proving Theorem 7, or local near unanimity term operations of arity $2k$, when proving Theorem 11, is sufficient for the proof.

▶ **Lemma 37.** *Let $(\mathbb{P}_1, Y)$ and $(\mathbb{P}_2, Y)$ be free fixing patterns with exactly one common vertex f, which is labeled by $x \notin Y$ and which is completely fixed in both patterns. For $i \in \{1, 2\}$ let $\mathbb{P}'_i$ be the pattern obtained from $\mathbb{P}_i$ by removing the fixing vertices and all the vertices labeled x (and all the incident faces). Let $\mathbb{Q}$ be the union of $\mathbb{P}_1$ and $\mathbb{P}_2$.*

If $(\mathbb{P}_i, Y)$, $i = 1, 2$ are f-realizable and $\mathbb{P}'_i$, $i = 1, 2$ are strongly realizable, then $(\mathbb{Q}, Y)$ is f-realizable.

Proof. Fix d, choose d'' so that each complete fixing $(\mathbb{S}, Z)$ of $\mathbb{P}'_1$ or $\mathbb{P}'_2$, which we will use in the proof, satisfies $d'' \geq z_{(\mathbb{S},Z)}(d + 1)$, and choose $d' \geq z_{(\mathbb{P}_i,Y)}(d'')$ for $i = 1, 2$.

Let $\phi : Y \to A$ be an evaluation of quality d' and denote $Y = \{y_1, \ldots, y_k\}$ (where variables can possibly repeat). For each $i \in \{1, 2\}$ and $j \in \{1, \ldots, k\}$ we construct a realization $\alpha_i^j : Q \to A$ of $\mathbb{Q}$ of quality d. The sought after quality-d extension α of ϕ will be obtained by applying a $2k$-ary (local) near unanimity operation to these realizations. In order to construct α_i^j we first extend ϕ to a realization β of $\mathbb{P}_i$ of quality d'' and define $\alpha_i^j(p) = \beta(p)$ for each $p \in \mathrm{dom}(\beta) = P_i$. Next, we extend the evaluation $\rho : \{x\} \cup Y \setminus \{y_j\} \to A$, defined by $\rho(x) = \beta(f)$ and $\rho(y) = \phi(y)$ else, to a quality-$(d + 1)$ realization γ of the complete $(\{x\} \cup Y \setminus \{y_j\})$-fixing of $\mathbb{P}'_{3-i}$ and define $\alpha_i^j(c) = \gamma(c)$ for each $c \in \mathrm{dom}(\gamma)$ (noting that ρ has quality d'' since β does and f is completely fixed in $\mathbb{P}_i$). Finally, for each face F of $\mathbb{Q}$ where α_i^j is not yet fully defined (this concerns fixing vertices of $\mathbb{P}_{3-i}$ labeled y_j) we use Lemma 34 and extend α_i^j so that it satisfies F with quality d. Now α_i^j satisfies all the faces of $\mathbb{Q}$ with quality d and agrees with ϕ on all of the fixing variables, except those from $\mathbb{P}_{3-i}$ labeled y_j. It follows that applying a $2k$-ary term operation to the α_i^j that satisfies the near unanimity condition for the set of components of the α_i^j gives an assignment of quality d (by Lemma 33) that extends ϕ, as required. ◀

▶ **Corollary 38.** *Let $(\mathbb{P}, Y)$ be a fixing pattern with two vertices $f_1 \neq f_2$ both labeled x and let n be a positive integer. Let $(\mathbb{Q}, Y)$ be the fixing pattern obtained from the disjoint union* *of n copies of $\mathbb{P}$ by identifying, for each $i \in \{1, \dots, n-1\}$, the vertex f_2 in the i-th copy with the vertex f_1 in the $(i+1)$-st copy. Let $\mathbb{P}'$ be the pattern obtained from $\mathbb{P}$ by removing the fixing vertices and all the vertices labeled x.*

If $(\mathbb{P}, Y)$ is f-realizable and $\mathbb{P}'$ is strongly realizable, then $(\mathbb{Q}, Y)$ is f-realizable.

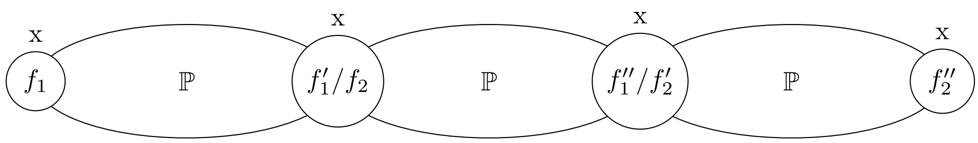

Figure 6 Pattern $\mathbb{Q}$ in Corollary 38.

Proof. The proof follows by induction from Lemma 37, noting that in each step if we remove vertices labeled x and fixing vertices from $\mathbb{Q}$, we get a pattern which is a disjoint union of strongly realizable patterns and is thus strongly realizable. ◀

The following lemma provides the second construction. The proof uses Corollary 38 (which requires a near unanimity term of arity $2k$ or local near unanimity term operations of arity $2k$) but the rest of the reasoning is based on the loop lemma stated in Theorem 26, for which a near unanimity term (or local near unanimity term operations) of any arity is sufficient.

▶ **Lemma 39.** *Let $(\mathbb{P}_1, Y)$ and $(\mathbb{P}_2, Y)$ be fixing patterns with a common inner face E and no other common vertices, such that both $\mathbb{P}_1$ and $\mathbb{P}_2$ are k-trees. For $i = 1, 2$ let f_i be a completely fixed inner vertex of $\mathbb{P}_i$ with label x such that $E \cup \{f_i\}$ is a face of $\mathbb{P}_i$. Let $\mathbb{Q}$ be the pattern obtained from the union of $\mathbb{P}_1$ and $\mathbb{P}_2$ by identifying vertices f_1 and f_2, and let $\mathbb{Q}'$ be the pattern obtained from $\mathbb{Q}$ (or $\mathbb{P}_1 \cup \mathbb{P}_2$) by removing the fixing vertices and all the vertices labeled x.*

If $(\mathbb{P}_1 \cup \mathbb{P}_2, Y)$ is f-realizable and $\mathbb{Q}'$ is strongly realizable, then $(\mathbb{Q}, Y)$ is f-realizable.

Proof. Let $r > 2$ be such that, in the case of proving Theorem 7, $\mathcal{V}$ has an r-ary near unanimity term, and in the case of proving Theorem 11, $\mathbf{A}$ has local near unanimity term operations of arity r (so $r = k+2$ works). Let $(\mathbb{Q}^{r-1}, Y)$ be the fixing pattern obtained by taking the disjoint union of $r-1$ copies of $\mathbb{P}_1 \cup \mathbb{P}_2$ and identifying the vertex f_2 in the i-th

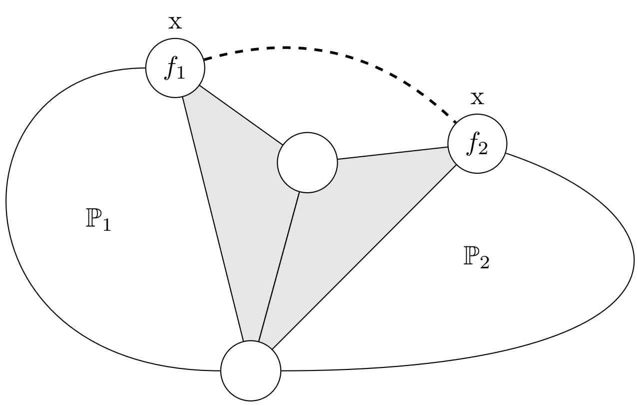

Figure 7 Patterns $\mathbb{P}_1 \cup \mathbb{P}_2$ and $\mathbb{Q}$ in Lemma 39

copy with the vertex f_1 in the $(i+1)$-first copy, for each $i \in \{1, \dots, r-1\}$. The pattern $(\mathbb{Q}^{r-1}, Y)$ is f-realizable by Corollary 38.

Fix d, choose d'' using Lemma 34 so that, for both $i \in \{1, 2\}$, every quality-d'' assignment $\alpha : E \cup \{f_i\} \to A$ extends to a quality-d realization of $\mathbb{P}_i$, and choose $d' \geq z_{(\mathbb{Q}^{r-1}, Y)}(d''+1)$.

Let $\phi : Y \to A$ be an evaluation of quality d'. Denote by B the set of all elements of $a \in A_x$ such that the evaluation $x \mapsto a$ has quality $d''+1$, denote by T the set of all the quality-d realizations β of $\mathbb{P}_1 \cup \mathbb{P}_2$ such that both $\{f_1\}$ and $\{f_2\}$ have quality $d''+1$ and both $E \cup \{f_1\}$ and $E \cup \{f_2\}$ have quality d'', and denote by $S \subseteq T$ the set of those $\beta \in T$ that extend ϕ. By a similar argument to that of Lemma 33, both T and S are subuniverses of $\prod_{p \in P_1 \cup P_2} \mathbf{A}_{v(p)}$. Using the near unanimity term of arity r (or local near unanimity term operations of arity r) S clearly locally r-absorbs T. The plan is to apply Theorem 28 to the binary relation $\mathrm{proj}_{f_1,f_2} S \subseteq B \times B$. If this binary relation contains a loop, then the corresponding $\alpha \in S$ satisfies $\alpha(f_1) = \alpha(f_2)$ and, therefore, we actually obtain a realization of $\mathbb{Q}$ of quality d, as required.

It remains to verify the assumptions of Theorem 28. By the choice of d', the pattern $\mathbb{Q}^{r-1}$ has a quality-$(d''+1)$ realization that extends ϕ. The images of copies of vertices f_1 and f_2 in such a realization yield a directed walk in $\mathrm{proj}_{f_1,f_2}(S)$ of length $r-1$. Next, since S locally r-absorbs T, then $\mathrm{proj}_{f_1,f_2}(S)$ locally r-absorbs $\mathrm{proj}_{f_1,f_2}(T)$, so it is enough to verify that the latter relation contains $=_B$ and $\mathrm{proj}_{f_1,f_2}(S)^{-1}$. For the first case, pick $b \in B$ and recall that the assignment $f_1 \mapsto b$ has quality $d''+1$ by the definition of B. We extend this assignment (using Lemma 34) to a quality d''-assignment $\alpha : E \cup \{f_1\} \to A$, define $\alpha(f_2) = \alpha(f_1)$, and extend α to a quality-d realization of $\mathbb{P}_1 \cup \mathbb{P}_2$. The obtained assignment witnesses $(b, b) \in \mathrm{proj}_{f_1,f_2}(T)$. Finally, to show that $\mathrm{proj}_{f_1,f_2}(T)$ contains $\mathrm{proj}_{f_1,f_2}(S)^{-1}$, consider any $(a, b) \in \mathrm{proj}_{f_1,f_2}(S)^{-1}$. By the definition of S, the pattern $\mathbb{P}_1 \cup \mathbb{P}_2$ has a realization α such that $\alpha(1) = b$, $\alpha(2) = a$, and both $E \cup \{f_1\}$ and $E \cup \{f_2\}$ have quality d''. We flip the values $\alpha(f_1)$ and $\alpha(f_2)$, restrict α to $E \cup \{f_1, f_2\}$ and extend this assignment using the choice of d'' to a realization of $\mathbb{P}_1 \cup \mathbb{P}_2$ of quality d, giving us $(a, b) \in \mathrm{proj}_{f_1,f_2}(T)$ and concluding the proof. ◀

D.3 Assembly

Lemma 36 and Lemma 39 enable us to prove that every k-tree is strongly realizable. We split the inductive proof of this fact into two lemmata.

▶ **Lemma 40.** *Let $1 \leq l \leq k$ and assume that every complete pattern with l vertices is strongly realizable. Then every l-tree is strongly realizable.*

Proof. It is enough to show that every complete l-tree is strongly realizable. However, for an inductive proof of this claim, it will be convenient to use more general l-trees, those that can be obtained from the empty pattern in n steps by taking the union of the already constructed pattern $\mathbb{S}$ with a complete pattern $\mathbb{C}$ on $l' \leq l$ vertices such that $S \cap C$ (where $0 \leq |S \cap C| < l'$) is a face in both patterns (with the same labelling in both patterns). The induction is primarily on n and secondarily on $|S \cap C|$. For $n = 1$ the claim follows from the assumption of the lemma. If $S \cap C = \emptyset$, then $\mathbb{S} \cup \mathbb{C}$ is a disjoint union and the claim follows by the inductive assumption and the assumption of the lemma.

Otherwise, take a fresh set Y of k-variables and let $(\mathbb{Q}, Y)$ be a complete Y-fixing of $\mathbb{S} \cup \mathbb{C}$. Pick a vertex in $S \cap C$, say vertex f_1 labeled x, let $\mathbb{C}'$ be the pattern obtained from $\mathbb{C}$ by renaming vertex f_1 to a fresh vertex f_2, let $(\mathbb{P}_1, Y)$ and $(\mathbb{P}_2, Y)$ be complete Y-fixings of $\mathbb{S}$ and $\mathbb{C}'$, respectively, and let $E = (S \cap C) \setminus \{f_1\}$. Note that this notation is consistent with the statement of Lemma 39: $\mathbb{Q}$ can be obtained from $\mathbb{P}_1 \cup \mathbb{P}_2$ by identifying vertices f_1

and f_2. To conclude the proof, we observe that the assumptions of Lemma 39 are satisfied. Indeed, $(\mathbb{P}_1 \cup \mathbb{P}_2, Y)$ is f-realizable by the inductive assumption (since it is a complete fixing of $\mathbb{S} \cup \mathbb{C}'$ for which $|S \cap C'| < |S \cap C|$) and $\mathbb{Q}'$ is strongly realizable since it is a subpattern of $\mathbb{S} \cup \mathbb{C}'$. ◀

▶ **Lemma 41.** *Let $1 < l \leq k+1$ and assume that every $(l-1)$-tree is strongly realizable. Then every complete pattern with l vertices is strongly realizable.*

Proof. We start with a complete vertex Y-fixing of a complete pattern with l vertices, which is f-realizable by Lemma 36, and add fixing faces one by one while preserving the f-realizability.

So, let $\mathbb{S}$ be an f-realizable Y-fixing of a complete pattern with l vertices and let $[E, Y']$ be such that $E = \{e_1, \ldots, e_{l'}\}$ is an inner face of $\mathbb{S}$ and $Y' \subseteq Y$ is a $(k - |E|)$-element set of variables. Our aim is to show that $\mathbb{S}$ plus the fixing face $[E, Y']$ is f-realizable. Let $(\mathbb{C}, Y)$ be the complete vertex Y-fixing of a complete pattern with the set of inner vertices $G = \{g_1, \ldots, g_{l'}\}$ (where g_i's are fresh vertices) labeled according to E (i.e., $v(g_i) = v(e_i)$ for each $i \in [l']$) with an additional fixing face $[G, Y']$. By Lemma 36, this fixing pattern is realizable. Let $(\mathbb{C}^i, Y)$, $i \in \{0, \ldots, l'\}$ be the fixing pattern obtained by renaming the vertices $g_1, \ldots, g_i$ to $e_1, \ldots, e_i$, respectively. The aim, reformulated, is to show that $(\mathbb{S} \cup \mathbb{C}^i, Y)$ is f-realizable for $i = l'$. We prove this claim by induction on i.

For $i = 0$ the union $\mathbb{S} \cup \mathbb{C}^i$ is disjoint, therefore the claim follows from the f-realizability of $\mathbb{S}$ and $\mathbb{C}^0 = \mathbb{C}$. For the induction step from i to $i+1$ we apply Lemma 39 with $\mathbb{P}_1 = \mathbb{S}$, $\mathbb{P}_2 = \mathbb{C}^i$, $f_1 = e_{i+1}$, and $f_2 = g_{i+1}$. Note that $(\mathbb{P}_1 \cup \mathbb{P}_2, Y)$ is f-realizable by the induction hypothesis and $\mathbb{Q}'$ is strongly realizable since it is an $(l-1)$-tree, so we can conclude that $(\mathbb{Q}, Y) = (\mathbb{S} \cup \mathbb{C}^{i+1}, Y)$ is f-realizable, finishing the proof. ◀

The following corollary is the core technical contribution of this section. Its proof follows by induction from the previous two lemmata.

▶ **Corollary 42.** *Every k-tree is strongly realizable.*

Armed with Corollary 42, we are ready to execute the idea outlined in the beginning of this section. For the purpose of the following theorem, we call an instance $\mathcal{I} = (V, \{\mathbf{A}_x \mid x \in V\}, \mathcal{C})$ a *weak k-instance* if it satisfies the running assumption, that is, $\mathcal{C} = \{(S, R_S) \mid S \subseteq V, |S| \leq k\}$ and, for any $S' \subseteq S$ such that $|S| \leq k$, the projection of R_S onto S' is contained in $R_{S'}$.

▶ **Theorem 43.** *Let $k \geq 2$ and $n \geq 0$ be integers. Then there exists $d = z(n,k)$ such that for any variety $\mathcal{V}$ with a $(k+2)$-ary near unanimity term, or any idempotent algebra $\mathbf{A}$ with local near unanimity term operations of arity $k+2$, any weak k-instance $\mathcal{I}$ of $\mathsf{CSP}(\mathcal{V})$ (or $\mathsf{CSP}(\mathbf{A})$) with at most n variables, and any at most k-element set of variables Y, every evaluation $\phi : Y \to A$ of quality d extends to a solution of $\mathcal{I}$.*

Proof. We prove the claim by induction on n. If $n \leq 1$, then the claim trivially holds with $d = 1$. Otherwise, we denote $d' = z(n-1, k)$ and pick a d greater than or equal to $z_{(\mathbb{T},Y)}(d')$ for every complete Y-fixing $(\mathbb{T}, Y)$ of a complete k-tree of depth d'.

Consider an instance $\mathcal{I}$ of $\mathsf{CSP}(\mathcal{V})$ (or $\mathsf{CSP}(\mathbf{A})$) and an evaluation $\phi : Y \to A$ of quality d. We define a new instance $\mathcal{I}' = (V', \{\mathbf{A}_x \mid x \in V'\}, \{(S, R'_S) \mid S \subseteq V, |S| \leq k\})$ by setting $V' = V \setminus Y$ and

$$R'_S = \{\rho_{|S} \mid \rho : Y \cup S \to A \text{ is a partial solution of } \mathcal{I} \text{ such that } \rho_{|Y} = \phi\}$$

Clearly, $\mathcal{I}'$ is a weak k-instance. We have chosen d so that, in the instance $\mathcal{I}$, the partial evaluation ϕ extends to a realization of the complete Y-fixing of a complete k-tree of depth

d' (the base can be chosen arbitrarily for the argument). This realization witnesses that, in the instance $\mathcal{I}'$, there exists an evaluation of quality d'. By the choice of d', any such evaluation extends to a solution θ of $\mathcal{I}'$. Now $\phi \cup \theta$ is a solution of $\mathcal{I}$, finishing the proof. ◀

To conclude, we state the parts of Theorem 7 and Theorem 11 that we set out to prove in this section as the following corollary. It directly follows from Lemma 31 and the previous theorem.

▶ **Corollary 44.** **1.** *If $\mathcal{V}$ is a variety that has a $(k+2)$-ary near unanimity term then every $(k, k+1)$-instance of the* CSP *over $\mathcal{V}$ is sensitive.*

2. *If $\mathbf{A}$ is an idempotent algebra that has local near unanimity term operations of arity $k+2$ then every $(k, k+1)$-instance of* $\mathsf{CSP}(\mathbf{A})$ *is sensitive.*